Solow Model

Production function exhibits **constant returns to scale** if $F(zK, zL) = zF(K, L)$. We can have $z = 1/L$ so that

$$Y = F(K, L) \quad \Longrightarrow \quad \frac{Y}{L} = F\left(\frac{K}{L}, 1\right).$$

- Y/L is output per worker and it is a function $F(K/L)$ of capital per worker K/L. (We can ignore that 1 above.)
- Let $y = Y/L$ and $k = K/L$. Then we can write $y = f(k)$.
- Let $C/L = c$ and $I/L = i$. It follows that $y = c + i$ if we ignore government and have a closed economy.
- Let s be the fraction of income that people save, so sy is the level of savings.
- Savings equals investment in a closed economy, so $i = sy$.
- Let γ be the depreciation rate. It measures the rate at which capital becomes worn down.
- Let n be the rate of growth of the labor force L.

The sum $\gamma + n$ captures the rate at which capital is being destroyed. If γ is high, then capital is of poor quality and falls apart quickly. If n is high, then there are more people using each piece of capital so it falls apart more quickly. Amount $(\gamma + n)k$ represents how much capital is *destroyed* in one period.

The amount of investment $sf(k) = sy = i$ represents how much capital is *created* in a period. So if $i > (\gamma + n)k$ this period, then you create more new capital than you destroy, so k will increase for next period. In math,

$$\Delta k = sf(k) - (\gamma + n)k.$$

Now notice that if $i > (\gamma + n)k$, then k will be larger in the next period. So then $(\gamma + n)k$ will be closer to i in the next period. It will keep growing in such a manner until $i = (\gamma + n)k^*$. The amount of capital being destroyed is exactly the same as the amount of capital being created, so there is no change in total capital in the next period. We call this the **steady-state** level of capital, k^*.

Vary the saving rate s to shift the sy curve and find a new steady-state level of capital k^* and a corresponding steady-state output $f(k^*)$. Let's make a graph of *only* steady-states. Plot all k^* and $f(k^*)$ for every possible level of s onto a graph along with the steady-state destruction line $(\gamma + n)k^* = sy = i$. There is a certain steady-state level of capital k^*_G that maximizes consumption. Since $c = y - i = f(k^*) - (\gamma + n)k^*$ at the steady-state, we can use calculus to show that c is maximized at MPK $= \gamma + n$. That is, c is maximized when the slope of the steady-state capital destruction line $(\gamma + n)k^*$ is tangent to the steady-state output $f(k^*)$, i.e. when MPK $= \gamma + n$. This is the **Golden Rule capital stock**.

Now suppose we care about how efficient labor is. Then we consider a sort of "effective labor force", $L \times E$, where E represents **worker efficiency**. We can rewrite everything in terms of $L \times E$, for instance now we have $f(k) = y = Y/(L \times E)$.

Let g be the growth rate of worker efficiency. It measures much more skillfully people do their jobs. *This is not technological improvement, just general skill improvement. Technological improvement is captured by another variable A that is hidden in $F(K, L)$.* If g is high, then people work "faster" and thus use each piece of capital more often. So now the capital destruction is $(\gamma + n + g)$, the steady-state condition is $(\gamma + n + g)k = sy = i$, and the Golden Rule level condition is MPK $= \gamma + n + g$.

At the steady state, $Y/(L \times E) = f(k^*)$. Since k^* is fixed by definition, it follows that $Y/(L \times E)$ is fixed. But we know that L is growing at a rate of n and E is growing at a rate of g, so it must be that Y is also growing at rate $n + g$ as well (so that the fraction $Y/(L \times E)$ does not change in value). So if we then consider just Y/L, we know that Y is growing at a faster rate than L is by the term g. So *income per laborer is increasing based on how more skilled the labor becomes.* This is analogous to an increase in MPL in the equation

$$Y = \text{MPK} \times K + \text{MPL} \times L.$$

Suppose MPK $> \gamma + n + g$. Then you can increase s to increase k^*, and a higher k^* implies a lower MPK because of diminishing marginal capital of $f(k)$ — higher k implies lower MPK and vice versa. Hopefully you will increase s until MPK $= \gamma + n + g$. Furthermore, higher s initially implies a lower c since $c = y - sy$. But as k^* makes its way to the Golden Rule level of capital k^*_G, eventually c will be at its maximum. So, provided you are saving too little now, you can start to save more now to maximize consumption later (try plotting the trajectory of c in this scenario). This is the other Golden Rule at work: "do unto others as you would have others to undo you." In other words, if you sacrifice and save more now, then you can increase the welfare of those in future generations. Presumably you would be grateful if your ancestors had sacrificed their own consumption to improve your welfare, so the Golden Rule says you should do the same thing.