Present Value

Present Value

Given constant discount rate r, a cash flow stream $(C_1, C_2, \ldots)$ has present value of

$$\begin{aligned}\mathrm{PV}(C_1, C_2, \ldots) &= \sum_{t=0}^{\infty} \mathrm{PV}(C_t) \\ &= \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + \ldots\end{aligned}$$

Including C_0, usually a cash outflow (i.e. payment) gives the **net present value (NPV)**. Note that r can be an interest rate on a bond or the cost of capital.

Perpetuity

A perpetuity that pays C each period has present value

$$PV = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \ldots = \frac{C}{r}.$$

Perpetuity with Growth

A perpetuity with growth rate has a present value of

$$\begin{aligned}\mathrm{PV} &= \frac{C_1}{(1+r)} + \frac{(1+g)C_1}{(1+r)^2} + \frac{(1+g)^2 C_1}{(1+r)^3} + \ldots \\ &= \frac{C_1}{r-g},\end{aligned}$$

where g is the growth rate of the payment each period. This formula is only valid if $g < r$.

Annuity

An annuity that pays C each period for T periods has a present value of

$$\begin{aligned}PV &= \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \ldots + \frac{C}{(1+r)^T} \\ &= \frac{C}{r}\left[1 - \left(\frac{1}{1+r}\right)^T\right].\end{aligned}$$

Annuity with Growth

An annuity with growth that pays C each period for T periods has a present value of

$$\begin{aligned}PV &= \frac{C}{(1+r)} + \frac{(1+g)C}{(1+r)^2} + \ldots + \frac{(1+g)^{T-1}C}{(1+r)^T} \\ &= \frac{C}{r-g}\left[1 - \left(\frac{1+g}{1+r}\right)^T\right],\end{aligned}$$

where g is the growth rate of the payment each period.

Coupon Bond

A coupon bond with face value FV, coupon payments C, and maturity of T periods, has present value of

$$\begin{aligned}PV &= \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \ldots + \frac{C}{(1+r)^T} + \frac{FV}{(1+r)^T} \\ &= \frac{C}{r}\left[1 - \left(\frac{1}{1+r}\right)^T\right] + \frac{FV}{(1+r)^T}.\end{aligned}$$

Resale Value

The resale value of a perpetuity after T periods is

$$RV = \frac{C}{r}\left(\frac{1}{1+r}\right)^T.$$

Converting Interest Rate Periods

For annual interest rate r_a, the corresponding monthly interest rate r_m is found by solving

$$(1+r_m)^{12} = 1 + r_a \quad \Longrightarrow \quad r_m = (1+r_a)^{1/12} - 1.$$

Weekly or daily interest rates are found by replacing 12 with 52 or 365, respectively. The corresponding five year interest rate r_5 is found by solving

$$(1+r_a)^5 = 1 + r_5 \quad \Longrightarrow \quad r_5 = (1+r_a)^5 - 1.$$

Capital Budgeting Rules

Net Present Value

Given constant discount rate r, a cash flow stream $(C_1, C_2, \ldots)$ with cost $C_0 \leq 0$ has net present value of

$$\begin{aligned}\mathrm{NPV} &= C_0 + \sum_{t=0}^{\infty} \mathrm{PV}_r(C_t) \\ &= C_0 + \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + \ldots\end{aligned}$$

If NPV > 0, then the stream of discounted cash flows exceeds the cost and thus the project is worth doing. Given a choice of projects, a firm will choose the project that gives the highest NPV.

Internal Rate of Return

For cash flow stream $(C_0, C_1, C_2, \ldots)$, the internal rate of return (IRR) is the number satisfying

$$0 = C_0 + \frac{C_1}{(1+\mathrm{IRR})} + \frac{C_2}{(1+\mathrm{IRR})^2} + \ldots$$

For this to be satisfied, we must have at least two C_i, C_j of opposite signs. We might find multiple positive solutions, or no positive solutions, in which case there is no well-defined IRR.

An investment project $(C_0, C_1, \ldots)$ with cost of capital r should be accepted when IRR is well-defined and greater than r. (Small r means discount future payments less.)

For investment financing, accept if IRR is well-defined and is less than r. (Large r means discount future costs more.)

Profitability Index

For cash flow stream $(C_0, C_1, C_2, \ldots)$, the profitability index is

$$\mathrm{PI} = \frac{\mathrm{PV}(C_1, C_2, \ldots)}{|C_0|}.$$

Intuitively, it is the ratio of present value of cash flow to the initial investment cost C_0. If the project generates a larger present value of cash flow than initial cost, then $PI > 1$ and the project is worth doing.