Replication and Put-Call Parity

Two portfolios with identical payoff diagrams must have equal prices by arbitrage argument – this is a pricing technique call **replication**.

Put-call parity is a specific type of replication, the form of which was given in lecture and discussions, via equation

$$P + S_0 = \mathrm{PV}(K) + C.$$

- K is the strike price on both options as well as the bond's face value.
- Portfolio 1: P is the price of the put option; S_0 is the price at which the share of stock was originally purchased.
- Portfolio 2: $\mathrm{PV}(K)$ is the present value of the bond's face value; C is the price of the call option.

For stocks with dividends, we augment the equation to

$$P + S_0 - \mathrm{PV}(\text{dividends}) = \mathrm{PV}(K) + C,$$

where PV(dividends) is the present value of any dividends.

Risk

Variance

The **variance** of random variable X is

$$\mathrm{Var}(X) = \sum_{i=1}^{n} p_i \big(X_i - E[X]\big)^2.$$

Risk Averse Preferences

An investor is **risk-averse** if she is (negatively) sensitive to variance. As an expositional device, think in terms of utility function

$$u(X) = E[X] - A\,\mathrm{Var}(X), \quad A \geq 0.$$

A is the *risk aversion parameter*. The point is this: utility is decreasing in variance (i.e risk) whenever $A > 0$. If $A = 0$, then we have risk-neutral investors.

Covariance and Correlation

Covariance measures *comovement* between random variables, that is, whether they both tend to move in the same direction from their respective means.

$$\mathrm{Cov}(X,Y) = \rho_{x,y}\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)},$$

where $\rho_{X,Y}$ is the **correlation coefficient** of X and Y. Note that $-1 \leq \rho_{X,Y} \leq 1$.

Weighted Portfolio

Consider risky assets X and Y with respective rates of return r_X and r_Y. Your portfolio consists of α_X proportion of asset X and α_Y proportion of asset Y. Hence the rate of return from your weighted portfolio W is

$$r_W = \alpha_X r_X + \alpha_Y r_Y, \quad \text{such that } \alpha_X + \alpha_Y = 1.$$

The expected rate of return for W is

$$E[r_W] = \alpha_X E[r_X] + \alpha_Y E[r_Y].$$

The variance of the rate of return for W is

$$\mathrm{Var}(r_W) = \alpha_X^2\,\mathrm{Var}(r_X) + \alpha_Y^2\,\mathrm{Var}(r_Y) + 2\alpha_x\alpha_y\,\mathrm{Cov}(r_X, r_Y).$$

Diversification

When X and Y are negatively correlated, it follows that $\mathrm{Cov}(r_X, r_Y) < 0$. Hence $\mathrm{Var}(r_W)$ is smaller than when $\mathrm{Cov}(r_X, r_Y) \geq 0$. This is a benefit of diversification – owning negatively correlated assets minimizes portfolio risk.

Intuitively, when one asset is doing poorly, the other is doing well, the effects cancel each other out to some extent.

Perfect Negative Correlation

If $\rho_{X,Y} = -1$, then

$$\mathrm{Var}(r_W) = \big[\alpha_X \mathrm{SD}(r_X) - \alpha_Y \mathrm{SD}(r_Y)\big]^2.$$

Therefore risk is zero when

$$\alpha_X \mathrm{SD}(r_X) = \alpha_Y \mathrm{SD}(r_Y).$$

If we are given standard deviations, then substituting $\alpha_Y = 1 - \alpha_X$ will allow us to solve for α_X.

Portfolio Choice

Terminal Value

Let $\widehat{W}$ denote the **terminal value** of an investment. Example: you buy a one year bond for \$100 and with rate of return of 10%. One year from now you receive that \$100 back in addition to \$10 in interest payments. So $\widehat{W} = \$110$.

In general, you have a choice of buying k different kinds of *risky* assets, and you spend a_i dollars on risky asset i.

Let $\hat{r}_i$ denote interest rate on risky asset i. Total return on risky assets is

$$\sum_{i=1}^{k} a_i(1 + \hat{r}_i).$$

Let W_0 denote initial wealth. Whatever is left over will be invested into a safe asset with interest rate r_f, giving

$$\left(W_0 - \sum_{i=1}^{k} a_i\right)(1 + r_f).$$

The sum of the two is the terminal wealth of the portfolio:

$$\widehat{W} = \left(W_0 - \sum_{i=1}^{k} a_i\right)(1 + r_f) + \sum_{i=1}^{k} a_i(1 + \hat{r}_i).$$

Portfolio Optimization Problem

You want to choose how much to spend on each asset in such a way that the terminal value of your portfolio maximizes your expected utility of holding such a portfolio:

$$\max_{a_1,\ldots,a_k} E\left[u\left(\widehat{W}\right)\right].$$

Take the partial derivative with respect to arbitrary asset purchase a_i to get the first order condition

$$E\left[u'(\widehat{W})(\hat{r}_i - r_f)\right] = 0.$$

Note that investing in risky asset i is optimal if and only if its expected interest rate is greater than that of the safe return, that is, $E[\hat{r}_i] > r_f$.