ECN 235B—Geromichalos, Licari, Suarez-Lledo (2007)

Adapted from Athanasios Geromichalos' lectures
William M Volckmann II

1 Introduction

This model will be very much like Lagos-Wright, except it will be an asset instead of money, which is a perfect substitute for money. The asset can be purchased at price ψ, and it pays off a dividend of R coconuts each period. Hence the value of buying the asset today is

$$\psi = \beta R + \beta^2 R + \ldots = \frac{\beta R}{1-\beta} = \psi^*,$$

where ψ^* is the **fundamental value** of the asset.

2 CM Value Functions

Buyers. Let a be current asset holdings. The buyer's value function is

$$W(a) = \max_{x,H,\hat{a}} u(x) - H + \beta V(\hat{a})$$

subject to $x + \psi\hat{a} = H + (\psi + R)a$. Notice that the benefit of holding an asset today is with the resale price ψ, similar to ϕ from the LW model, but also the coconut dividend of R.

Solve the constraint for H, plug it into the value function, and then replace x with optimal x^* to get

$$\begin{aligned} W(a) &= u(x^*) - x^* + (\psi + R)a + \max_{\hat{a}} \{-\psi\hat{a} + \beta V(\hat{a})\} \\ &= \Lambda + (\psi + R)a. \end{aligned}$$

Sellers. In the Lagos-Wright model, sellers didn't bring any money into the decentralized market; they traded it all for the general good while in the central market because they have no use for money otherwise. Same story here, I guess. So the seller value function is

$$W^s(a) = \max_{x,H}\{u(x) - H + \beta V^s(0)\},$$

subject to $x = H + (\psi + R)a$. Using the same chain of logic as before, we can write this as

$$\begin{aligned} W^s(a) &= u(x^*) - x^* + (\psi + R)a + \beta V^s(0)\} \\ &= \Lambda^s + (\psi + R)a. \end{aligned}$$

3 DM Bargaining

Hey, let's *not* do Nash bargaining because it's gross. All hail Kalai.

3.1 Surpluses

Buyer Surplus. Right, so when trade goes down, the buyer receives q of the special good and pays d_a for it. Hence buyer surplus is

$$\begin{aligned} BS &= u(q) + W(a - d_a) - W(a) \\ &= u(q) + \Lambda + (\psi + R)(a - d_a) - \Lambda - (\psi + R)a \\ &= u(q) - (\psi + R)d_a. \end{aligned}$$

They gain utility of consumption but lose the value of the asset they give up. Intuitive, oui?

Seller Surplus. The seller pays the cost of producing the special good, $-q$, but continues on with d_a of the asset. Hence seller surplus is

$$\begin{aligned} SS &= -q + W^s(d_a) - W^s(0) \\ &= -q + \Lambda + (\psi + R)d_a - \Lambda + (\psi + R)0 \\ &= -q + (\psi + R)d_a. \end{aligned}$$

Again, intuition is straightforward.

3.2 Kalai

The Kalai bargaining problem is

$$\max_{q,d_a} u(q) - (\psi + R)d_a \quad \text{s.t.} \quad u(q) - (\psi + R)d_a = \frac{\theta}{1-\theta}[-q + (\psi + R)d_a],$$

as well as the asset constraint $d_a \leq a$. Mess around with the Kalai constraint a bit and you'll end up with

$$(\psi + R)d_a = (1-\theta)u(q) + \theta q \equiv z(q).$$

Plug this into the objective function and we have

$$\max_{q} \theta[u(q) - q].$$

Case 2: Optimality. Assume a is huge. Then we can choose optimal q^*. Hence the Kalai constraint gives

$$d_a = \frac{(1-\theta)u(q^*) + \theta q^*}{(\psi + R)} = \frac{z(q^*)}{(\psi + R)} \equiv a^*,$$

where a^* now denotes the required amount to afford the optimal transaction.

Case 1: Asset Constrained. Now let's assume that $d_a = a \leq a^*$. The best option is to achieve the largest transaction possible, which requires spending everything so that $d_a = a$. This corresponds to transaction of

$$\tilde{q}(a) = \{q : (\psi + R)a = z(q)\}.$$

Bargaining Solution. Hence the solution to the bargaining problem is

$$\big(q(a), d_a(a)\big) = \begin{cases} (q^*, a^*) & \text{if } a > a^*, \\ (\tilde{q}(a), a) & \text{if } a \leq a^*. \end{cases}$$

4 DM Value Functions

There's a σ probability of running into a seller who accepts the asset. Hence the buyer's value function is

$$\begin{aligned}
V(a) &= \sigma\left[u\big(q(a)\big) + W\big(a - d_a(a)\big)\right] + (1-\sigma)W(a) \\
&= \sigma\left[u\big(q(a)\big) + \Lambda + (\psi + R)(a - d_a(a))\big)\right] + (1-\sigma)[\Lambda + (\psi+R)a] \\
&= \sigma\left[u\big(q(a)\big) - (\psi+R)d_a(a)\right] + \Lambda + (\psi+R)a \\
&= \sigma\left[u\big(q(a)\big) - (\psi+R)d_a(a)\right] + W(a).
\end{aligned}$$

Now consider the CM value function,

$$W(a) = u(x^*) - x^* + (\psi+R)a + \max_{\hat{a}}\left\{-\psi\hat{a} + \beta V(\hat{a})\right\}.$$

Define the expression in the max operator to be $J(\hat{a})$. With our expression for $V(a)$, we can write

$$\begin{aligned}
J(\hat{a}) &= -\psi\hat{a} + \beta V(\hat{a}) \\
&= -\psi\hat{a} + \beta\sigma u\big(q(\hat{a})\big) - \beta\sigma(\hat{\psi}+R)d_a(\hat{a}) + \beta W(\hat{a}) \\
&= -\psi\hat{a} + \beta\sigma u\big(q(\hat{a})\big) - \beta\sigma(\hat{\psi}+R)d_a(\hat{a}) + \beta\Lambda + \beta(\hat{\psi}+R)\hat{a} \\
&= \left[-\psi + \beta(\hat{\psi}+R)\right]\hat{a} + \beta\sigma\left[u\big(q(\hat{a})\big) - (\hat{\psi}+R)d_a(\hat{a})\right],
\end{aligned}$$

where all terms without $\hat{a}$ are omitted because they won't affect the max operator. The first term is the net cost of carrying the asset, and the first term is the discounted expected buyer surplus that one additional asset will buy.

Quick thought experiment. Suppose $\sigma = 0$ so that no one accepts the asset as payment and thus it has no liquidity value – its entire value comes from being a store of value. We already know that the net cost of carrying the asset can't be negative in a steady state because otherwise demand for the asset would be infinite; and hence we conclude that $\beta(\psi + R) \geq \psi$. On the other hand, since the only worth the asset has is via a store of value, the cost of carrying the thing can't be positive either because otherwise no one would buy the thing; and hence $\beta(\psi + R) \leq \psi$. The only steady-state equilibrium must then satisfy $\beta(\psi + R) = \psi$, which in turn implies that

$$\psi = \frac{\beta R}{1-\beta} = \psi^*.$$

Makes sense – the asset has no use as a currency, so its value must be equal to its fundamental value, i.e. its sum of discounted dividends.

5 One-Period Assets

As a benchmark case, let's consider an asset that lives for only one period, which is to say, you receive R in the next period and then the asset is junk and has no resale value. This means the ψ term on the RHS the value function constraints drops out, i.e. $(\psi + R)$ turns into R, and therefore

$$\begin{aligned}
\big(q(a), d_a(a)\big) &= \begin{cases} (q^*, a^*) & \text{if } a > a^*, \\ (\tilde{q}(a), a) & \text{if } a \le a^*, \end{cases} \\
\frac{z(q^*)}{R} &\equiv a^*, \\
J(\hat{a}) &= [-\psi + \beta R]\,\hat{a} + \beta\sigma\left[u\big(q(\hat{a})\big) - R d_a(\hat{a})\right].
\end{aligned}$$

5.1 Asset Demand

Case 1. Suppose $\psi = \beta R$, so that the cost of carrying the asset is zero. Then $\psi = \psi^*$. The buyer wants to carry at least $\hat{a} = a^*$ so that the optimal q^* can be had. But there's no cost to holding more than that.

Case 2. Suppose $\psi > \beta R$ so that there's a positive cost of carrying the asset. Then there's no reason to bring more than $\hat{a} = a^*$, so $q = \tilde{q}(a)$ and $d_a(a) = a$. Then we are to maximize

$$J(\hat{a}) = [-\psi + \beta R]\,\hat{a} + \beta\sigma\left[u\big(\tilde{q}(\hat{a})\big) - Ra\right].$$

The first order condition gives

$$\psi = \beta R + \beta\sigma\left[u'\big(\tilde{q}(\hat{a})\big)\tilde{q}'(\hat{a}) - R\right].$$

We can clean this up a little bit by noting that

$$R\hat{a} = z\big(q(\hat{a})\big) \quad \Longrightarrow \quad \tilde{q}'(\hat{a}) = \frac{R}{z'\big(\tilde{q}(\hat{a})\big)},$$

and therefore

$$\psi = \beta R \left(1 + \sigma \left[\frac{u'\left(\tilde{q}(\hat{a})\right)}{z'\left(\tilde{q}(\hat{a})\right)} - 1 \right] \right).$$

We'll want to evaluate this expression to determine the shape of the demand function. The two constants on the outside don't amount to anything, so let's restrict our analysis to

$$e(q) \equiv 1 + \sigma \left[\frac{u'\left(\tilde{q}(\hat{a})\right)}{z'\left(\tilde{q}(\hat{a})\right)} - 1 \right].$$

First, let's see what happens at q^*. We know that $u'(q^*) = 1$. Furthermore,

$$z(q) \equiv (1-\theta)u(q) + \theta q \quad \Longrightarrow \quad z'(q) = (1-\theta)u'(q) + \theta,$$

which evaluated at q^* gives $z'(q^*) = 1$. It follows that $e(q^*) = 1$ and thus $\psi = \beta R$. This is consistent with case 1, so the universe need not yet implode.

Now let's analyze at $q = 0$. We'd have

$$\begin{aligned} e(0) &= 1 + \sigma \left[\frac{u'(0)}{z'(0)} - 1 \right] \\ &= 1 + \sigma \left[\frac{u'(0)}{(1-\theta)u'(0) + \theta} - 1 \right] \\ &= 1 + \sigma \left[\frac{1}{(1-\theta) + \frac{\theta}{u'(0)}} - 1 \right] \\ &= 1 + \frac{\sigma\theta}{1-\theta}. \end{aligned}$$

From this we conclude that

$$q = 0 \quad \Longrightarrow \quad \psi = \beta R \left[1 + \frac{\sigma\theta}{1-\theta} \right].$$

Finally, we need to show that $e(q)$ is monotonically decreasing. To do so, we really only need to focus on

$$\frac{u'(q)}{(1-\theta)u'(q) + \theta}.$$

We'll use the quotient rule, which will give a squared denominator and hence is irrelevant as far as the sign is concerned. So focus only on the numerator, which gives

$$[(1-\theta)u'(q) + \theta]u''(q) - u'(q)(1-\theta)u''(q) = \theta u''(q),$$

which is unambiguously negative by preference assumptions.

5.2 Asset Supply and Equilibrium

The equilibrium price of the asset will be determined by the supply of the asset. If the supply is less than $z(q^*)/R \equiv a^*$, then the asset is scarce enough that the price of the asset will include a *liquidity premium* above the fundamental value βR. The traded quantity will be less than $\tilde{q}(\hat{a}) = q^*$.

If supply is larger than $z(q^*)/R \equiv a^*$, then scarcity is not an issue and the price of the asset is only its fundamental value. The traded quantity will be optimal q^*, even if asset holdings exceed a^*.

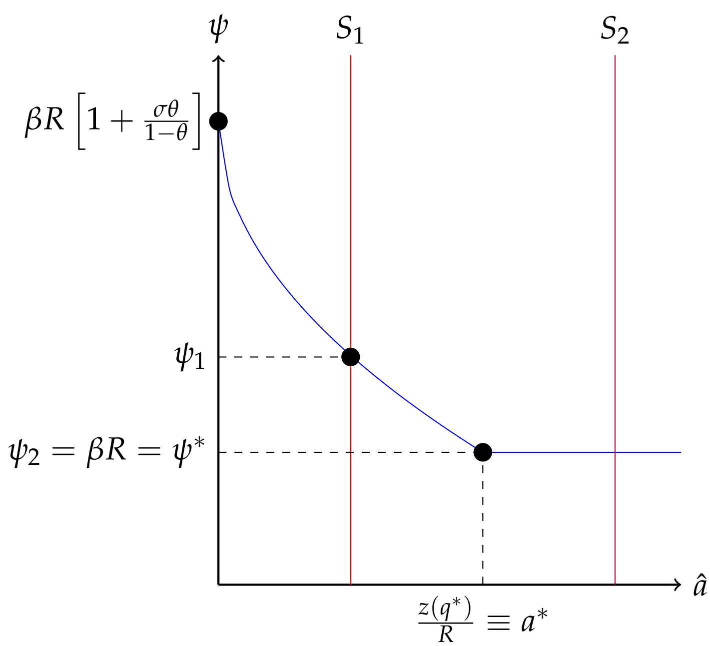

Figure 1: With asset supply $S_2 \geq a^*$, liquidity of the asset is not a concern and hence the price of the asset is only its fundamental value, βR. With asset supply is $S_2 < a^*$, the asset possesses a liquidity premium due to its scarcity, $\psi_1 - \psi^*$.

6 Long-Lived Assets

Okay, now assume that the asset has a stream of dividends and thus retains its resale value. Then we are back to the case where

$$\big(q(a), d_a(a)\big) = \begin{cases} (q^*, a^*) & \text{if } a > a^*, \\ (\tilde{q}(a), a) & \text{if } a \leq a^*, \end{cases}$$
$$\frac{z(q^*)}{\psi + R} \equiv a^*,$$
$$J(\hat{a}) = \left[-\psi + \beta(\hat{\psi} + R)\right] \hat{a} + \beta\sigma \left[u\big(q(\hat{a})\big) - (\hat{\psi} + R) d_a(\hat{a})\right].$$

6.1 Asset Demand

Let's focus on the steady-state where $\psi = \hat{\psi}$, giving

$$J(\hat{a}) = \left[-\psi + \beta(\psi + R)\right] \hat{a} + \beta\sigma \left[u\big(q(\hat{a})\big) - (\psi + R) d_a(\hat{a})\right].$$

Case 1. Suppose there is no net cost to holding assets so that

$$\psi = \frac{\beta R}{1 - \beta} = \psi^*.$$

Hence the bargaining solution gives

$$\begin{aligned} a^* &\equiv \frac{z(q^*)}{\psi + R} \\ &= \frac{(1 - \beta) z(q^*)}{R}. \end{aligned}$$

Okay, so no cost of holding the asset means the optimal amount needed to purchase first best is as given above. And again, holding more than this is not irrational since there is no cost to holding more. Any asset supply at or above a^* will result in optimal q^*.

Case 2. Suppose there is a strictly positive cost of holding assets so that $\psi > \beta(\psi + R)$. This implies that the buyer will never hold more than a^*, and hence will always pay $d_a = a$. To find optimal asset demand, we take the first order condition of $J(\hat{a})$ with respect to $\hat{a}$ to get

$$\psi = \beta(\psi + R) + \beta\sigma \left[u'\big(\tilde{q}(\hat{a})\big)\tilde{q}'(\hat{a}) - (\psi + R)\right].$$

This is becoming repetitive. We know that $(\psi + R)\hat{a} = z\big(\tilde{q}(\hat{a})\big)$, and therefore

$$\psi + R = z'\big(\tilde{q}(\hat{a})\big)\tilde{q}'(\hat{a}) \quad \Longrightarrow \quad \tilde{q}'(\hat{a}) = \frac{\psi + R}{z'\big(\tilde{q}(\hat{a})\big)},$$

from which it follows that

$$\begin{aligned}\psi &= \beta(\psi + R)\left(1 + \sigma\left[\frac{u'\big(\tilde{q}(\hat{a})\big)}{z'\big(\tilde{q}(\hat{a})\big)} - 1\right]\right) \\ &= \beta(\psi + R)e\big(\tilde{q}(\hat{a})\big).\end{aligned}$$

This is nice because we already know everything we need to know about $e(q)$. In particular, we know that $e(q^*) = 1$, and hence $\psi = \beta R/(1-\beta) = \psi^*$, as it should be. We also know that

$$e(0) = 1 + \frac{\sigma\theta}{1-\theta} \quad \Longrightarrow \quad \psi = \beta(\psi + R)\left[1 + \frac{\sigma\theta}{1-\theta}\right].$$

Finally, we know that $e'(q) < 0$, and hence we can conclude that the RHS falls as q increases. We know there is a one-to-one monotonic relationship between q and a, so we know it also falls as a increases. The result is pretty much the same as before.

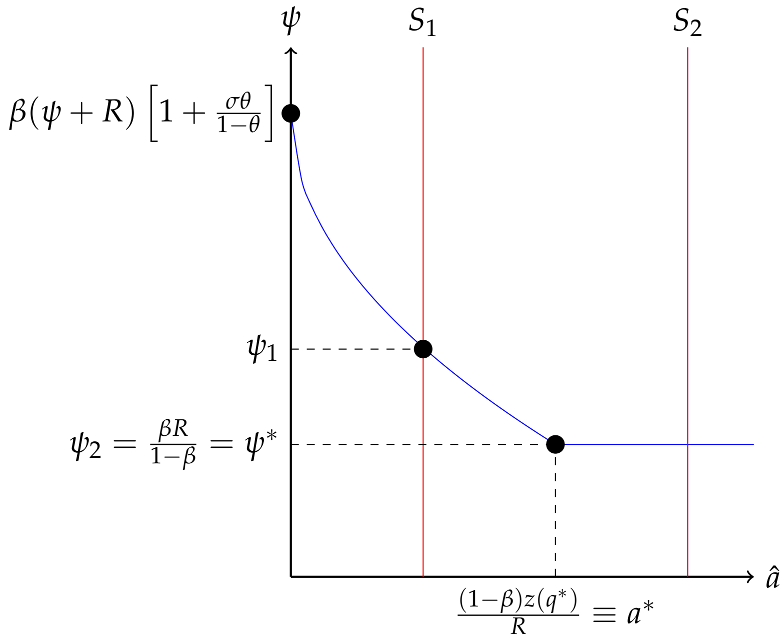

Figure 2: With asset supply $S_2 \geq a^*$, liquidity of the asset is not a concern and hence the price of the asset is only its fundamental value, βR. With asset supply is $S_2 < a^*$, the asset possesses a liquidity premium due to its scarcity, $\psi_1 - \psi^*$.

It can be easily ascertained from Figure 2 that for asset supply lower than a^*, the price is decreasing in A and quantity traded is increasing in A.

7 General Model

Now let's include both money and the asset. Woo.

7.1 Central Market Value Functions

The CM value function for the buyer becomes

$$W(a, m) = \max_{x,H,\hat{a},\hat{m}} \{u(x) - H + \beta V(\hat{a}, \hat{m})\}$$

subject to $x + \phi\hat{m} + \psi\hat{a} = H + (\psi + R)a + \psi m + T$. Solve the constraint for H, plug it into the value function, plug in optimal x^*, and we get

$$\begin{aligned} W(a, m) &= u(x^*) - x^* + (\psi + R)a + \psi m + T + \max_{\hat{a},\hat{m}}\{-\phi\hat{m} - \psi\hat{a} + \beta V(\hat{a}, \hat{m})\} \\ &= \Lambda + (\psi + R)a + \psi m. \end{aligned}$$

Nothing fancy about the seller, so we can just take it for granted that $W^s(a,m) = \lambda + (\psi + R)a + \psi m$. Once again, the seller has no need for holding either assets nor cash and hence will carry zero into the DM.

7.2 Bargaining

Buyer Surplus. The buyer walks in with a and m. They consume q for utility $u(q)$. Hence buyer surplus is

$$\begin{aligned} BS &= u(q) + W(a - d_a, m - d_m) - W(a, m) \\ &= u(q) + \Lambda + (\psi + R)(a - d_a) + \psi(m - d_m) - \Lambda - (\psi + R)a - \psi m \\ &= u(q) - (\psi + R)d_a - \psi d_m. \end{aligned}$$

Buyer surplus is utility gained minus the lost value of the money and asset.

Seller Surplus. Seller has no currency. They lose q as the cost of producing q. Hence seller surplus is

$$\begin{aligned} SS &= -q + W(d_a, d_m) - W(0,0) \\ &= -q + \Lambda + (\psi + R)d_a + \phi d_m - \Lambda \\ &= -q + (\psi + R)d_a + \phi d_m. \end{aligned}$$

Seller surplus is the benefit from carrying new money and assets minus the cost of producing things.

Kalai. The bargaining objective is to solve

$$\max_{d_a, d_m, q} u(q) - (\psi + R)d_a - \psi d_m$$

subject to the constraints

$$\begin{aligned} u(q) - (\psi + R)d_a - \psi d_m &= \frac{\theta}{1-\theta}\left[-q + (\psi + R)d_a + \phi d_m\right], \\ d_a &\leq a, \\ d_m &\leq m. \end{aligned}$$

Take the Kalai constraint and push some things around to get

$$(\psi + R)d_a + \phi d_m = (1-\theta)u(q) + \theta q \equiv z(q).$$

Oh hey it's the same thing. Kalai bargaining sure is reliable, oui? Plug this into the objective function and we get

$$\max_q \theta[u(q) - q] \quad \text{s.t.} \quad d_a \leq a,\ d_m \leq m.$$

Case 1: Optimality. Suppose that a and/or m are huge. Doesn't matter which, just as long as the sum of their worth is enough to purchase q^*. Then, um, q^* will be purchased. It follows that $(\psi + R)d_a + \phi d_m = z(q^*)$. We can't determine the precise allocation of assets vs. bonds, but that doesn't really matter here anyway – it's total purchasing power that matters. To that end, define

$$\pi(a, m) \equiv (\psi + R)a + \phi m$$

to be the **aggregate liquid portfolio**. In this case, $\pi(a, m) = z(q^*)$ is the purchasing power required for first best.

Case 2: Funding Constrained. Now suppose that total purchasing power is not enough to afford beyond q^*, so that $\pi(a, m) \leq z(q^*)$. Then all funds will be spent, i.e. $d_a = a$ and $d_m = m$. Hence the quantity traded will be

$$\tilde{q}(a, m) \equiv \{q : \pi(a, m) = z(q)\}.$$

Bargaining Solution. So the solution to the Kalai bargaining problem is

$$\begin{cases} \text{if } \pi(a, m) > z(q^*), \quad q = q^* \text{ and } \pi(d_a, d_m) = z(q^*), \\ \text{if } \pi(a, m) \leq z(q^*), \quad q = \tilde{q}(a, m) \text{ and } \pi(a, m) = z\big(\tilde{q}(a, m)\big). \end{cases}$$

In words. If the buyer is carrying enough wealth to afford the first best, then they buy the first best by paying any feasible combination of assets and money that exactly affords it. If the buyer is not carrying enough wealth, then they spend everything they have and receive $q = \tilde{q}(a, m)$.

7.3 Rates of Return

We have defined $\pi = \phi m + (\psi + R)a$ to be the aggregate liquid portfolio, representing total wealth of liquid assets. If $\pi \geq z(q^*)$, then q^* is purchased for some undetermined combination of assets satisfying $\phi m + (\psi + R)a = z(q^*)$. If $\pi < z(q^*)$, then everything is spent, i.e. $d_m = m$ and $d_a = a$, and $\tilde{q}(a, m) : \{q : \pi = z(q)\} < q^*$ is purchased. In case you're wondering, $z(q)$ is what you'd expect it to be at this point.

You can verify this if you'd like, but it's a process that's highly similar to what was seen in previous models. So let's skip to the point and note that

$$\begin{aligned} J(\hat{a}, \hat{m}) &= \left[-\psi + \beta\hat{\psi}\right]\hat{a} + \left[-\phi + \beta\hat{\phi}\right]\hat{m} \\ &= \beta\sigma\left[u\big(q(\hat{a}, \hat{m}) - (\hat{\psi} + R)d_a(\hat{a}, \hat{m}) - \hat{\phi}d_m(\hat{a}, \hat{m})\big)\right] \end{aligned}$$

The interpretation is as similar to the usual: the net cost of carrying assets, the net cost of carrying money, and the discounted expected buyer surplus that having more wealth will buy.

Case 1. Suppose there are enough assets floating around so that $A \geq A^* = (1 - \beta)z(q^*)/R$. In other words, all liquidity needs are met through assets alone. Assets fully serve as both a medium of exchange and a store of value, and hence money is not essential – introducing money would not increase welfare in this case.

That does not, however, imply that money could not circulate. Indeed, if the rate of return on money is exactly the same as that of assets, then the two are perfect substitutes. Recall that the asset will be at its fundamental value $\beta R/(1 - \beta)$ if it is sufficiently liquid. Hence the steady-state rates of return are, respectively,

$$\rho_a = \frac{(\hat{\psi} + R) - \psi}{\psi} \stackrel{ss}{=} \frac{R}{\psi^*} = \frac{R}{\beta R/(1 - \beta)} = \frac{1}{\beta} - 1 > 0,$$

$$\rho_m = \frac{\hat{\phi} - \phi}{\phi} \stackrel{ss}{=} \frac{\hat{\phi}\hat{M}}{\phi M(1 + \mu)} - 1 = \frac{1}{1 + \mu} - 1.$$

Thus the rates of return are equalized when $\mu = \beta - 1 < 0$, which is true when the Friedman rule is satisfied. Makes sense – zero interest rate means the cost of carrying money is its lowest. Woo.

Case 2. Now suppose that $A < A^*$ so that assets alone cannot manage to establish a transaction of q^*. Perhaps introducing money will pick up the slack. Let's find out.

We'll take the FOC of the objective function when $A < A^*$, i.e. when we are to the left of q^* and hence on the binding branch of the bargaining solution. So the objective function is

$$\begin{aligned} J(\hat{a}, \hat{m}) = & \left[-\psi + \beta\hat{\psi}\right]\hat{a} + \left[-\phi + \beta\hat{\phi}\right]\hat{m} \\ & + \beta\sigma\left[u\big(\tilde{q}(\hat{a}, \hat{m}\big) - (\hat{\psi} + R)\hat{a} - \hat{\phi}m\right] \end{aligned}$$

The FOC with respect to $\hat{m}$ and $\hat{a}$ are, respectively,

$$\begin{aligned} \phi &= \beta\hat{\phi} + \beta\sigma\left[u'(\tilde{q})\frac{d\tilde{q}}{d\hat{m}} - \hat{\phi}\right], \\ \psi &= \beta(\hat{\psi} + R) + \beta\sigma\left[u'(\tilde{q})\frac{d\tilde{q}}{d\hat{a}} - (\hat{\psi} + R)\right]. \end{aligned}$$

To find the derivatives, take the $(\psi + R)a + \phi m = z(\tilde{q})$ equation and totally differentiate:

$$\begin{aligned} \phi dm = z'(\tilde{q})d\tilde{q} \quad &\Longrightarrow \quad \frac{d\tilde{q}}{dm} = \frac{\phi}{z'(\tilde{q})}, \\ (\psi + R)da = z'(\tilde{q})d\tilde{q} \quad &\Longrightarrow \quad \frac{d\tilde{q}}{da} = \frac{\psi + R}{z'(\tilde{q})}. \end{aligned}$$

Thus we can write the FOCs – money and asset demand equations, respectively – as

$$\begin{aligned} \phi &= \beta\hat{\phi}\left(1 + \sigma\left[\frac{u'(\tilde{q})}{z'(\tilde{q})} - 1\right]\right), \\ \psi &= \beta(\hat{\psi} + R)\left(1 + \sigma\left[\frac{u'(\tilde{q})}{z'(\tilde{q})} - 1\right]\right). \end{aligned}$$

Once again, let $e(q)$ denote the expression in the big parentheses. Now divide the two out and you get

$$\begin{aligned} \frac{\phi}{\psi} = \frac{\hat{\phi}}{\hat{\psi} + R} \quad &\Longrightarrow \quad \frac{\phi M(1+\mu)}{\psi} = \frac{\hat{\phi}\hat{M}}{\hat{\psi} + R} \\ &\stackrel{ss}{\Longrightarrow} \quad \frac{1+\mu}{\psi} = \frac{1}{\psi + R} \\ &\Longrightarrow \quad \psi = -\frac{R(1+\mu)}{\mu}. \end{aligned}$$

So we can write $\psi = \psi(\mu)$, which illustrates how monetary policy can affect the price of an asset. (Asset supply is embedded in this result since we are assuming that $A < A^*$.)

So hey, let's think about how. If μ goes up, then inflation goes up, so money becomes more expensive. Therefore demand for assets goes up and the price of assets goes up accordingly: $\psi'(\mu) > 0$. And down goes the rate of return. It's easy enough to show that I'll omit it.

7.4 Equilibrium

Notice that in the steady state, the money demand equation and asset demand equations give, respectively,

$$\frac{1+\mu}{\beta} = e(q),$$
$$\psi = \beta(\psi + R)e(q).$$

Let z denote real money balances. A steady-state equilibrium is a triplet (z, ψ, q) such that

(a) $\frac{1+\mu}{\beta} = e(q)$,

(b) $\psi = \beta(\psi + R)e(q)$,

(c) $z + (\psi + R)A = z(q)$.

The lowest possible value we can have for ψ is its fundamental value ψ^*. As μ increases, the price of assets increases. It's worth asking if there's an upperbound for μ, call it $\bar{\mu}$, that still allows money to be worth holding for any reason – if μ and thus inflation are really high, then the rate of return on money goes to hell and only assets will be held. At least, that's the intuition. But let's go into more detail.

If $A > A^*$, then money only has a use when the Friedman rule is adhered to, i.e. when $\mu = \beta - 1 = \bar{\mu}$. Otherwise

$$\bar{\mu} \equiv \{\mu : \psi(\mu) = \bar{\psi}\},$$

where $\bar{\psi}$ is the asset price in a world without money. Let's prove it.

Suppose otherwise – that is, suppose there exists a monetary equilibrium, so that $z' > 0$, while $\mu' > \bar{\mu}$. This implies that $\psi' > \bar{\psi}$ since ψ is increasing in μ. Notice that

$$\begin{aligned} z(q') &= z' + (\psi' + R)A \\ &> (\bar{\psi} + R)A \\ &= z(\bar{q}). \end{aligned}$$

This is the same $z(q)$ function of old, so we know that it is increasing in q, and hence we can conclude that $q' > \bar{q}$.

Okay, so $\mu' > \bar{\mu}$ implies that $q' > \bar{q}$. But uh, the money demand equation and the fact that $e'(q) < 0$ implies that q should *decrease* with μ, that is, it implies $q' < \bar{q}$ because $\mu' > \bar{\mu}$. We have a contradiction. So we have shown that there exists an upperbound to money growth/inflation that allows for a monetary equilibrium, and that upperbound is $\bar{\mu} \equiv \{\mu : \psi(\mu) = \bar{\psi}\}$.

7.5 Summary

Before proceeding, notice that $\bar{\mu}$ is a function of A when $A \leq A^*$. If $A = A^*$, then $\bar{\mu} = \beta - 1$. As A gets smaller – there are fewer assets in the market – then money plays a larger role so inflation/money growth can be a little bit higher. In other words, $\bar{\mu}'(A) < 0$.

Case A. If $A \geq A^*$, then $\psi = \psi^*$ and $q = q^*$. Money could circulate ($z > 0$) if $\mu = \beta - 1$, but it won't be essential. If there is no money, then $\bar{\psi}$ solves

$$\begin{aligned} z(\bar{q}) &= (\bar{\psi} + R)A, \\ \bar{\psi} &= \beta(\bar{\psi} + R)e(\bar{q}). \end{aligned}$$

Case B. If $A > A^*$, then a monetary equilibrium will exist if

(a) $\mu \in [\beta - 1, \bar{\mu}]$ where $\bar{\mu} = \bar{\mu}(A)$, $\bar{\mu}'(A) < 0$, and $\bar{\mu}(A^*) = \beta - 1$.

(b) For any $\mu \in [\beta - 1, \bar{\mu}]$, there exists a unique steady-state equilibrium (z, ψ, q) such that

- z is decreasing in μ,
- q is decreasing in μ,
- ψ is increasing in μ,
- ρ_A is decreasing in μ.