ECN 235B—Lagos-Wright Model

Adapted from Athanasios Geromichalos' lectures
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1 Kiyotaki-Wright

The Kiyotaki-Wright (KW) model is a monetary search model consisting of producers, commodity traders, and money traders.

- **Producers.** If an agent has no good, then they go produce one with Poisson rate a. Then they become a commodity trader.
- **Commodity Traders.** As a commodity trader, they either try to find a double coincidence of wants—trader A has what trader B wants, trader B has what trader A wants, so they just swap and go back to production. Or they try to find someone who wants their good who has money, possibly trading it for that (if money isn't completely worthless), after which they become a money trader.
- **Money Traders.** As a money trader, they try to find someone who has a good they want and will accept money. Then they go back to production.

I'm not going to go through the details (because it was said to be optional), but it turns out that there are three possible equilibria depending on beliefs about the likelihood of money being accepted—a pure exchange equilibrium, a pure money equilibrium, and a mixed equilibrium.

The KW model is nice and everything, but it's largely intractable. The **Lagos-Wright** model addresses this intractability.

2 Lagos-Wright

Every period will have two subperiods. The first subperiod is a KW decentralized market (DM). The second subperiod is a Walrasian centralized market (CM) in which there is no need for money (but money can still be exchanged if desired).

We will assume that preferences are quasilinear because wealth effects are jerks and quasilinear preferences have no wealth effects. This means that the optimal choice is unaffected by the number of assets on hand while making a decision in any given period.

The Lagos-Wright model is built around the following environment.

- The discount rate is β.
- There are two types of agents, buyers B and sellers S. They remain the same type for life.
- There are two commodities. The *general good* is traded and consumed in the centralized market. Everyone likes the general good.
- The *special good* is produced by sellers during the DM, which they do not consume and hence try to trade. Not everyone likes the special good, which necessitates the double coincidence of wants or money trading.
- Buyers work in the CM and consume the general good. They buy the special good in the DM and consume it.
- Sellers also work the CM and consume the general good. In the DM, they produce and sell their special good for money.
- We will assume that utility $u(\cdot)$ is twice continuously differentiable and furthermore that $u' > 0$, $u'' < 0$, and $u'(0) = \infty$.
- We will assume that there exists some optimal x^* such that $u'(x^*) = 1$.
- We will assume that there exists some optimal q^* such that $u'(q^*) = 1$.
- Goods are nonstorable.
- Money is storable and fiat.
- The central bank chooses a constant rate of money growth μ such that $M_{t+1} = (1 + \mu)M_t$.
- A lump sum T is transferred to buyers in the CM.
- Buyers meet sellers in the DM with probability $\sigma \in [0, 1]$.
- Buyers and sellers bargain in the DM, where θ is buyer bargaining power.

The only feasible trades during the DM are the barter of the special good and the exchange of special good for money. In the CM, the only feasible trades involve the general good and money.

3 Centralized Market Value Functions

3.1 The Buyer

Let $V(\cdot)$ be the buyer's value function for the DM. Then the buyer's value function for the CM is

$$W(m) = \max_{x,H,m'} \{u(x) - H + \beta V(m')\}.$$

They begin in the CM with m dollars. They want to choose how much of the general good x to buy and consume, how many hours H to work, and how much money m' to carry over into the upcoming DM.

The buyer's budget constraint is given by

$$x + \phi m' = H \cdot 1 + \phi m + T,$$

where ϕ is the amount of the general good the agent has to give up for one unit of money. So the amount of consumption plus the real cost of carrying money m' into the next period equals the earnings from hours worked (one general good per hour) plus the real value of money on hand plus whatever payment is transferred.

Let's first solve the budget constraint for H and plug it into the value function. Then we've assumed that there is some optimal x^*, so let's plug that into the value function as well. Doing so allows us to write

$$W(m) = u(x^*) - x^* + \phi m + T + \max_{m'} \{ -\phi m' + \beta V(m')\}.$$

Here we can see that there are no wealth effects—current money holdings m do not affect the max operator. Furthermore, $W(m)$ is linear in m, so we can simplify and write

$$W(m) = \Lambda + \phi m, \tag{1}$$

where Λ is all that other junk.

3.2 The Seller

This case is less interesting. First, sellers choose how much to work and how much of the general good to consume. However, sellers do not want to take money into the DM because it does them no good—as per their namesake, they're only selling stuff in the DM—so they'll trade all their money holdings to buyers in the CM in exchange for the general good, which is then consumed. In this case, we'll have

$$W^s(m) = \max_{x,H}\{u(x) - H + \beta V^s(0)\}$$

subject to $x = H + \phi m$. (The reason sellers come into the CM with money is that they just sold stuff in the DM.)

Let's again solve the budget constraint for H, plug it into the Bellman equation, and then impose the optimal x^* so that the Bellman equation can be written as

$$\begin{aligned} W^s(m) &= u(x^*) - x^* + \beta V^s(0) + \phi m \\ &= \Lambda^s + \phi m, \end{aligned}$$

where Λ^s is all of the other junk.

4 Decentralized Market Bargaining

We want to consider the typical meeting between a buyer who carries m dollars into the DM and the seller who carries zero. We'll use Kalai bargaining for now. (I'll hit up Nash later—it's more complicated.) Let q be the quantity of the special good purchased and d be the number of dollars spent. Let BS be the buyer surplus and SS be the seller surplus. Then we are to solve

$$\max_{q,d} \text{BS} \quad \text{s.t.} \quad \text{BS} = \frac{\theta}{1-\theta}\text{SS} \quad \text{and} \quad d \leq m,$$

the first constraint being the *Kalai constriant*, the second being the *cash constraint*.

4.1 Buyer Surplus

The buyer purchases q units of the special good for d dollars, after which q is consumed. The buyer continues on in the DM with $m - d$ dollars and leaves state $W(m)$. We can write this as

$$\begin{aligned} BS &= u(q) + W(m-d) - W(m) \\ &= u(q) + \Lambda + \phi m - \Lambda - \phi d - \phi m \\ &= u(q) - \phi d. \end{aligned}$$

Now you see why the Λ grouping came in handy. This expression is intuitive: the buyer surplus is the utility gained minus the real value of the dollars spent.

4.2 Seller Surplus

Recalling that the seller begins in the DM with no money, the seller surplus from a trade is

$$\begin{aligned} \text{SS} &= -q + W^s(d) - W^s(0) \\ &= -q + \Lambda^s + \phi d - \Lambda^s \phi \cdot 0 \\ &= -q + \phi d. \end{aligned}$$

They gain the real value of the dollars they receive, but lose the cost of producing q special goods, which is $c(q) = q$. Again, pretty intuitive.

4.3 Bargaining Solution

And therefore the bargaining problem becomes

$$\begin{aligned} \max_{q,d} \quad & u(q) - \phi d \\ \text{s.t.} \quad & u(q) - \phi d = \frac{\theta}{1-\theta}(\phi d - q), \\ & d \leq m. \end{aligned}$$

From the Kalai constraint, it follows that

$$\begin{aligned} [1-\theta][u(q) - \phi d] &= \theta[\phi d - q] \\ \implies \quad \phi d &= \theta q + (1-\theta)u(q) \\ &= z_K(q). \end{aligned}$$

We can use this to rewrite the objective function as

$$\max_q u(q) - \theta q - (1-\theta)u(q) = \max_q \theta[u(q) - q],$$

which is still subject to $d \leq m$. There are two cases with respect to money holdings that we must consider.

Case 1: Optimality. If m is large enough, then the cash constraint is irrelevant. In this case, there is nothing preventing the bargaining solution from selecting the optimal q^*. Using the Kalai constraint, this means the buyer will pay

$$\phi d = z_K(q^*) \implies \phi d = z_K(q^*) = \phi m^*.$$

In other words, let m^* be the amount of cash needed to be able to afford the optimal q^*. If the buyer has $m \geq m^*$ in cash, then they can pay $d^* = m^*$ and the solution is optimal.

Case 2: Cash Constrained. If $m \leq m^*$, then the buyer will spend as much as they can, i.e. $d = m \leq m^*$. The amount traded will be determined by the Kalai constraint, in particular it satisfies

$$\tilde{q}_K(m) = \{q : \phi m = z_K(q)\}.$$

If they brought exactly m^*, then this is optimal. But anything less than m^* a suboptimal outcome since $\tilde{q}_K(m) < q^*$.

Kalai Solution. We can now define the solution of Kalai bargaining as

$$\big(q(m), d(m)\big) = \begin{cases} (q^*, m^*) & \text{if } m > m^*, \\ (\tilde{q}_K(m), m) & \text{if } m \leq m^*. \end{cases}$$

5 Decentralized Market Value Functions

The seller is boring and we already know everything we need to know about them. For the buyer, we have

$$V(m) = \sigma \left[u\big(q(m)\big) + W\big(m - d(m)\big)\right] + (1-\sigma)W(m),$$

where $q(m)$ and $d(m)$ arise from bargaining. The first term represents the probability that a buyer meets a seller, gaining utility for consuming $q(m)$ for $d(m)$ dollars and then continuing to the CM with $m - d(m)$ dollars. The second term represents not being able to find a seller and thus continuing on to the CM with all of their money. Recalling that

$W(m) = \Lambda + \phi m$, we can rewrite this as

$$\begin{aligned} V(m) &= \sigma \left[u\big(q(m)\big) + \Lambda + \phi m - \phi d(m)\right] + (1-\sigma)(\Lambda + \phi m) \\ &= \sigma \left[u\big(q(m)\big) + \Lambda + \phi m - \phi d(m)\right] + \Lambda + \phi m - \sigma(\Lambda + \phi m) \\ &= \sigma \left[u\big(q(m)\big) - \phi d(m)\right] + W(m). \end{aligned}$$

Let's use this expression for $V(m)$ in the the original Bellman equation,

$$W(m) = u(x^*) - x^* + \phi m + T + \max_{m'}\{-\phi m' + \beta V(m')\}.$$

Define the expression inside the max operator to be

$$\begin{aligned} J(m') &\equiv -\phi m' + \beta V(m') \\ &= -\phi m' + \beta \left(\sigma[u\big(q(m')\big) - \phi' d(m')] + W(m')\right) \\ &= -\phi m' + \beta \left(\sigma[u\big(q(m')\big) - \phi' d(m')] + \Lambda' + \phi' m'\right) \\ &= -\phi m' + \beta\phi' m' + \beta\Lambda' + \beta\sigma \left[u\big(q(m')\big) - \phi' d(m')\right] \\ &= (-\phi + \beta\phi')m' + \beta\sigma \left[u\big(q(m')\big) - \phi' d(m')\right]. \end{aligned}$$

The first term captures the net cost of carrying a unit of money from one period to another. The second term is the discounted expected buyer surplus that one additional unit of money will buy. The $\beta\Lambda'$ term has been omitted entirely since it is not a function of m' and therefore doesn't affect our subsequent analysis.

6 Money Growth

6.1 Money Demand

Let's assume that $\mu > \beta - 1$, which in turn implies that $i > 0$. We will only consider the Friedman rule as a limiting case.

So far we've denoted m^* as the amount of money required to buy the first-best, and the buyer never needs to bring more than that. Let's further assume that we will always be in the binding branch of the bargaining solution, i.e. the buyer will never actually bring more than m^*. This allows us to write

$$J(m') \equiv (-\phi + \beta\phi')m' + \beta\sigma \left\{u\big(\tilde{q}_K(m')\big) - \phi' m'\right\}. \tag{2}$$

Claim 1. *In any equilibrium, $\phi \geq \beta\phi'$.*

Proof. The cost of carrying money cannot be negative, although it could be zero. If $\phi < \beta\phi'$, then you'd carry $m' = \infty$. This cannot be the case. So it must be the case that $(\beta\phi' - \phi)m' \leq 0$. □

Claim 2. *A nonmonetary equilibrium where $\phi = \phi' = 0$ always exists.*

Proof. Since money has no value and no price, it's entirely arbitrary what the price turns out to be—it could be anything, in particular, zero. □

In a monetary equilibrium, $m' > 0$. Taking the first order condition of equation (2) with respect to m', we get

$$\phi = \beta\phi' + \beta\sigma \left\{ u'\big(\tilde{q}_K(m')\big)\tilde{q}_K'(m') - \phi' \right\}.$$

Recall that $\tilde{q}_K(m')$ solves $\phi m' = z_K(q)$, and therefore $\phi m = z_K\big(\tilde{q}_K(m')\big)$. Taking the derivative with respect to m', it follows that

$$\phi = z_K'\big(\tilde{q}_K(m)\big)\tilde{q}'(m) \quad \Longrightarrow \quad \tilde{q}'(m') = \frac{\phi'}{z_K'\big(\tilde{q}_K(m')\big)}.$$

Plug this into the first order condition for

$$\begin{aligned} \phi &= \beta\phi' + \beta\sigma \left(\frac{u'\big(\tilde{q}_K(m')\big)}{z_K'\big(\tilde{q}_K(m')\big)}\phi' - \phi' \right) \\ &= \beta\phi' \left[1 + \sigma \left(\frac{u'\big(\tilde{q}_K(m')\big)}{z_K'\big(\tilde{q}_K(m')\big)} - 1 \right) \right] \\ \Longrightarrow \quad \frac{\phi}{\beta\phi'} - 1 &= \sigma \left[\frac{u'\big(\tilde{q}_K(m')\big)}{z_K'\big(\tilde{q}_K(m')\big)} - 1 \right]. \end{aligned} \tag{3}$$

Equation (3) is the demand for money.

6.2 Steady State

Let's focus on the steady state where real balances are equal across periods, i.e. where $\phi M = \phi' M'$, and hence $\phi M = \phi'(1+\mu)M$. Then we can write

$$\begin{aligned}\frac{\phi}{\beta\phi'} - 1 &= \frac{M(1+\mu)\phi}{M(1+\mu)\beta\phi'} - 1 \\ &= \frac{M\phi(1+\mu)}{M'\phi'\beta} - 1 \\ &= \frac{1+\mu}{\beta} - 1.\end{aligned}$$

Now ask yourself: what price p would you pay in order to receive a coconut one period from now? Well, your discount rate β, and therefore $\beta(1+r) = 1$, which in turn implies that $\beta = 1/(1+r)$. Also, convince yourself that in the steady state, inflation will be the money growth rate μ. The Fisher equation says that $1+i = (1+r)(1+\pi)$, but using the information just provided,

$$i = \frac{1+\mu}{\beta} - 1.$$

Oh hey, that's the LHS of the money demand equation. So we can write the demand for money as

$$i = \sigma \left[\frac{u'\left(\tilde{q}_K(m')\right)}{z_K'\left(\tilde{q}_K(m')\right)} - 1 \right]. \tag{4}$$

Great, we have an equation that gives equilibrium q for any interest rate i.

Definition 1. A **steady state monetary equilibrium** is a list of objects (q, Z), where q is the amount of the special good in a typical trade (we're interested in how close to q^* it is) and $Z = \phi M > 0$ are real money balances such that:

(a) q is pinned down via the interest rate through money demand

$$i = \sigma \left[\frac{u'\left(\tilde{q}_K(m')\right)}{z_K'\left(\tilde{q}_K(m')\right)} - 1 \right],$$

(b) Real money balances are pinned down via q through Kalai constraint

$$\phi m = Z = z_K(q) = \theta q + (1-\theta)u(q).$$

7 Comparative Statics

Noting that $z'(q) = \theta + (1-\theta)u'(q)$, write the RHS of the money demand function as

$$R_K(q) = \sigma \left[\frac{u'(q)}{\theta + (1-\theta)u'(q)} - 1 \right]$$

At optimal q^*, we assume that $u'(q^*) = 1$. And therefore

$$R_K(q^*) = \sigma \left[\frac{1}{\theta + (1-\theta)} - 1 \right] = 0.$$

This implies that the optimal q^* requires an interest rate of $i = 0$, i.e. the Friedman rule prevails.

Now rewrite $R_K(q)$ again a little by dividing $u'(q)$ for

$$R_K(q) = \sigma \left[\frac{1}{\frac{\theta}{u'(q)} + (1-\theta)} - 1 \right].$$

As $q \to 0$ we arrive at the Inada condition $u'(0) = \infty$ and thus it follows that

$$R_K(0) = \sigma \left[\frac{1}{1-\theta} - 1 \right] = \frac{\sigma\theta}{1-\theta}.$$

Graphically, we now have our two intercepts.

We still need to figure out whether the thing is sloping upwards or downwards. To do that, use the quotient rule and note that its denominator is squared, and thus positive, and thus we don't care about it. The numerator ends up being $\theta u''(q) < 0$ due to concavity of utility. Hence we know that the function is strictly downward sloping, as shown in Figure 1.

Notice that higher i corresponds to lower q. This makes sense—higher interest rate increases the opportunity cost of holding money, hence fewer transactions of the special good. Also notice that higher σ, which implies a higher chance of making a transaction, shifts $R_K(q)$ up and hence leads to more transactions and more money holding.

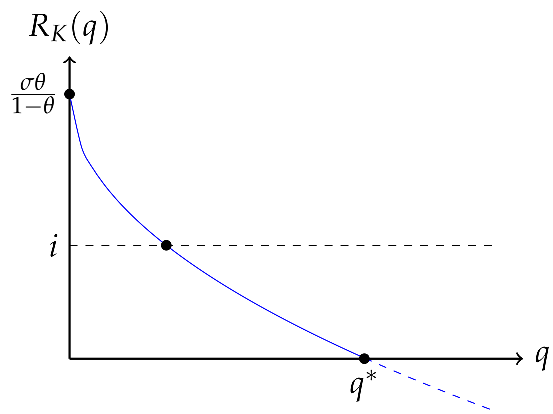

Figure 1: Equilibrium comes at the intersection of i and $R_K(q)$. Only when the Friedman rule is follows does the economy achieve its first best. When $i \geq \sigma\theta/(1-\theta)$, the interest rate is so high than no one wants to hold money and hence $q = 0$.

8 Nash Solution

8.1 Bargaining

The Nash bargaining equation is

$$\max_{q,d} [u(q) - \phi d]^{\theta} [\phi d - q]^{1-\theta} \quad \text{s.t.} \quad d \leq m.$$

Logging it up makes things easier. And let's incorporate the cash constraint as the Lagrangian

$$L = \theta \log \left(u(q) - \phi d\right) + (1-\theta) \log(\phi d - q) - \lambda(d - m).$$

The first order condition with respect to q and m, as well as the complementary slackness conditions, are

$$\begin{aligned} &\frac{\theta u'(q)}{u(q) - \phi d} = \frac{1-\theta}{\phi d - q}, \\ -&\frac{\theta \phi}{u(q) - \phi d} + \frac{\phi(1-\theta)}{\phi d - q} = \lambda, \\ &\lambda(d - m) = 0. \end{aligned}$$

Let's once again consider the nonbinding and binding cases separately.

Case 1: Optimality. This is the case where m is large enough that we don't have to care about whether the buyer can afford the best. This implies that q^* is traded. The nonbinding cash constraint means $d < m$ and therefore $\lambda = 0$ by complementary slackness. So our first order conditions reduce to

$$\frac{\theta u'(q^*)}{u(q^*) - \phi d} = \frac{1-\theta}{\phi d - q^*},$$
$$\frac{\theta \phi}{u(q^*) - \phi d} = \frac{\phi(1-\theta)}{\phi d - q^*}.$$

Add the two to, noting that $u'(q^*) = 1$ by assumption, to get

$$\frac{\theta + \theta\phi}{u(q^*) - \phi d} = \frac{1-\theta+\phi(1-\theta)}{\phi d - q^*} \quad \Longrightarrow \quad d = \frac{(1-\theta)u(q^*) + \theta q^*}{\phi} = m^*.$$

Oh hey, this is the same optimal money holding from Kalai bargaining. Woo.

Case 2: Cash Constrained. Now for the binding constraint where $d = m$, which technically leaves the possibility of $\lambda = 0$, but we'll ignore that case. So $d = m$ and $\lambda \neq 0$. Solving the first of the FOCs gives

$$\frac{\theta u'(q)}{u(q) - \phi m} = \frac{1-\theta}{\phi m - q} \quad \Longrightarrow \quad \phi m = \frac{(1-\theta)u(q) + \theta u'(q) q}{(1-\theta) + \theta u'(q)} = z_N(q).$$

So the quantity of special good traded will be

$$\tilde{q}_N(m) = \{q : \phi m = z_N(q)\}.$$

Nash Solution. We can now define the solution of Nash bargaining as

$$\big(q(m), d(m)\big) = \begin{cases} (q^*, m^*) & \text{if } m > m^*, \\ (\tilde{q}_N(m), m) & \text{if } m \leq m^*. \end{cases}$$

8.2 Money Demand and Comparative Statics

In the Kalai case we mostly kept things general in terms of the $z(\cdot)$ function, so we can continue at the point with basically the same money demand equation,

$$i = \sigma \left[\frac{u'\big(\tilde{q}_N(m')\big)}{z'_N\big(\tilde{q}_N(m')\big)} - 1 \right].$$

So the only difference notationally is the change from subscript K to N.

It will, however, be messier to consider the comparative statics because $z_N(q)$ is a lot messier than $z_K(q)$. Nonetheless, let's trudge on. We'll need to know what $z_N'(q)$ is at some point, so let's just get that out of the way right now. Actually, *you* get that out of the way now because it's tedious. Use the quotient rule and simplify the numerator until you get

$$z_N'(q) = \frac{\theta(1-\theta)u''(q)[q-u(q)] + u'(q)[1-\theta+\theta u'(q)]}{[1-\theta+\theta u'(q)]^2}.$$

Let $R_N(q)$ by the RHS of this money demand equation. For optimal q^*, we have $u'(q^*) = 1$ and

$$\begin{aligned} z_N'(q^*) &= \frac{\theta(1-\theta)u''(q^*)[q^*-u(q^*)] + u'(q^*)[1-\theta+\theta u'(q^*)]}{[1-\theta+\theta u'(q^*)]^2} \\ &= \theta(1-\theta)u''(q^*)[q^*-u(q^*)] + 1, \end{aligned}$$

from which it follows that

$$\begin{aligned} R_N(q^*) &= \sigma\left[\frac{1}{\theta(1-\theta)u''(q^*)[q^*-u(q^*)]+1} - 1\right] \\ &= \sigma\left[\frac{-\theta(1-\theta)u''(q^*)[q^*-u(q^*)]}{\theta(1-\theta)u''(q^*)[q^*-u(q^*)]+1}\right]. \end{aligned}$$

We need to establish something about $q^* - u(q^*)$, and we can do so by drawing the marginal cost curve, assumed to be q, and the utility function as determined by the Inada conditions.

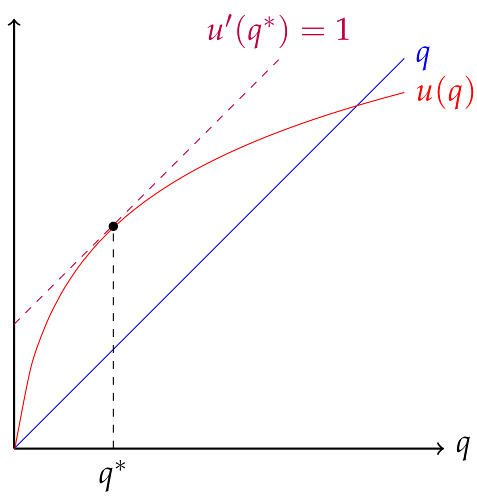

Figure 2: Optimal q^* comes with tangency of slope 1, putting it above marginal cost q^*.

It is clear that $u(q^*) > q^*$, and hence we conclude that $q^* - u(q^*) < 0$. The numerator then is the product of three negatives and is therefore negative. The denominator is the product of two negatives plus a positive number, and hence is positive. So $R_N(q^*) < 0$. That doesn't sound good; in fact, it implies that the optimal q^* can only be achieved with a *negative* interest rate.

Now for the limiting case where $q \to 0$. It is easy to see right away that the numerator will go to zero because $u(0) = \infty$. To analyze $z'_N(0)$, write

$$\begin{aligned} z'_N(q) &= \frac{\theta(1-\theta)u''(q)[q-u(q)] + u'(q)[1-\theta+\theta u'(q)]}{[1-\theta+\theta u'(q)]^2} \\ &= \frac{\theta(1-\theta)u''(q)[q-u(q)]}{[1-\theta+\theta u'(q)]^2} + \frac{1}{\frac{1}{u'(q)} - \frac{\theta}{u'(q)} + \theta}. \end{aligned}$$

The first term drops out as the denominator blows up to infinity, as will the two terms in the second denominator. So we're left with $z'_N(0) = 1/\theta$. It follows that $R_N(0) = \sigma[\theta\infty - 1] = \infty$. It asymptotes.

It turns out that showing how the two extremes connect is incredibly difficult; Randy had to write an entirely separate paper showing that $R'_N(q) < 0$. Well whatever, we can now superimpose the two bargaining solutions.

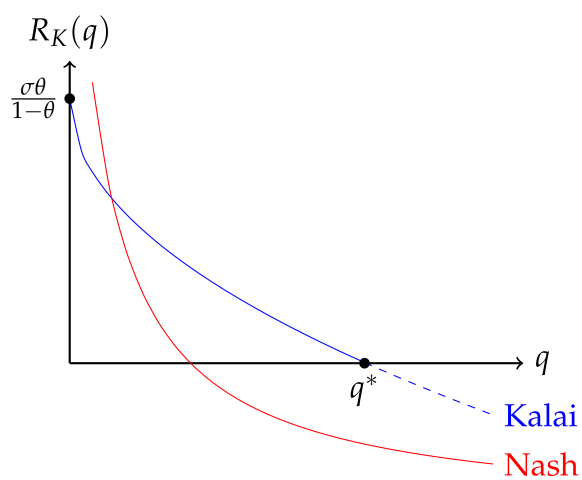

Figure 3: Nash bargaining asymptotes as $q \to 0$, and it also never achieves q^* for any positive interest rate.

9 Conclusions

Proposition 1. *With Nash bargaining, a unique monetary equilibrium always exists and q_N is decreasing in i.*

Proposition 2. *With Kalai bargaining, existence of a monetary equilibrium is not guaranteed: a monetary equilibrium requires $i < \sigma\theta/(1-\theta)$. When such an equilibrium exists, q_K is decreasing in i.*

Proposition 3. *Under both bargaining schemes, welfare maximization requires the Friedman rule where $i = 0$. In Kalai bargaining, this gives the first best result $q_K = q^*$. In Nash bargaining, this gives second-best $q_N < q^*$.*

The reason Nash gives the weird result is that is it not *proportional* like Kalai bargaining is. If you plot total surplus and buyer surpluses for both Kalai and Nash, the Nash surplus is shifted to the left a bit whereas Kalai is directly underneath total surplus.

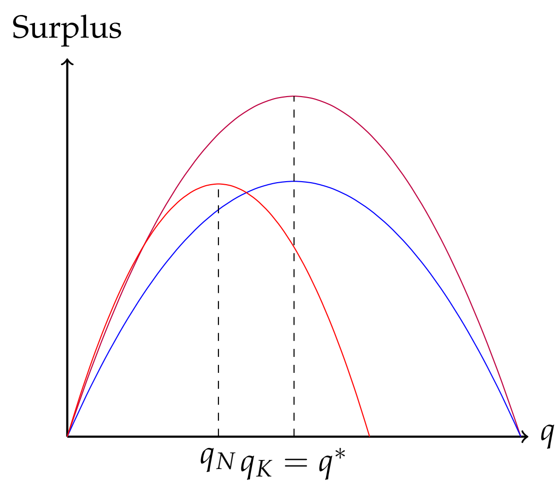

Figure 4: Purple is total surplus, blue is Kalai buyer surplus, and red is Nash buyer surplus. For Kalai, buyer surplus is maximized at the same q^* where total surplus is maximized. For Nash, buyer surplus is maximized at $q_N < q^*$.