ECN 235B—Duffie, et al (2007)

Adapted from Athanasios Geromichalos' lectures
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Most trade – around 85% of trade – occurs in over-the-counter markets. Basically everything there is traded except for stocks. So hey, let's model it.

1 Environment

There are two types of asset holders, or "investors" – low types and high types. Low types value the asset little, high types value it a lot. The idea is that we want low-type owners to sell their asset to high-type non-owners. There will be search frictions in the two matching, however. To that end, there will also be "market makers" functioning as financial intermediaries: they will hook up low-type owners with high-type nonowners... for a fee.

- Time is continuous (it's a finance paper).
- r is the discount rate.
- Investors can hold $\{0, 1\}$ of the asset.
- ℓ_o are low-type asset owners, and ℓ_n are low-type asset non-owners.
- h_o are high-type asset owners, and h_n are high-type asset non-owners.
- μ_{ij} is the measure of each type of investor in each state, e.g. μ_{hn} or $\mu_{\ell o}$. These should sum to 1.
- Asset supply is fixed and is given by $s < 1$. Note that $\mu_{\ell o} + \mu_{ho} = s$.

There will be a periodic *taste shock* for which some low types will change into high types and vice versa. Specifically, an h type becomes an ℓ type rate λ_d; call this a "down shock." An ℓ type becomes an h type at rate λ_u; call this an "up shock."

2 Poisson Rates

There is a Poisson rate λ at which investors meet at random. Therefore the probability that someone in group D independently contacts group C is given by $\lambda\mu_C\mu_D$. The probability that someone in group C independently contacts group D is given by $\lambda\mu_D\mu_C$. Thus the total meeting rate between the two groups is $2\lambda\mu_C\mu_D$.

The measure of market makers (MM) is 1. They have access to a competitive and frictionless (i.e. Walrasian) interdealer market. MM don't hold any assets: they simply find another MM that has a match for their client. Let ρ be the arrival rate of investors to a match maker.

When two investors match each other, $q \in (0, 1)$ is the bargaining power of the seller. When a MM and an investor match, $z \in (0, 1)$ is the bargaining power of the MM.

3 Equilibrium Accounting

Let's consider how the measure of each type of agent will change from one period to the next.

First consider low-type owners. If they find someone to trade with, then they will change into ℓ_n; this occurs at rate $2\lambda\mu_{\ell o}\mu_{hn}$. If they find a MM, then they change into ℓ_n, but the rate is a little trickier. This is because there might be a different number of high type sellers than low type buyers. If there are more buyers than sellers, then trade volume is determined by the number of sellers; and vice versa. Hence the rate at which sellers meet a MM who find a buyer is $\rho \min\{\mu_{\ell o}, \mu_{hn}\}$. Measure $\lambda_u\mu_{\ell o}$ get hit by a taste shock which turns them into high type owners. But measure $\lambda_d\mu_{ho}$ high type owners become low type owners. Thus the total change in the measure of low type owners is

$$\dot{\mu}_{\ell o} = -2\lambda\mu_{\ell o}\mu_{hn} - \rho \min\{\mu_{\ell o}, \mu_{hn}\} - \lambda_u\mu_{\ell o} + \lambda_d\mu_{ho}. \tag{1}$$

The story is similar for each type of agent, so I'll omit the narrative for each and simply jot down the equations.

$$\dot{\mu}_{hn} = -2\lambda\mu_{\ell o}\mu_{hn} - \rho \min\{\mu_{\ell o}, \mu_{hn}\} - \lambda_d\mu_{hn} + \lambda_u\mu_{\ell n}, \tag{2}$$

$$\dot{\mu}_{ho} = 2\lambda\mu_{\ell o}\mu_{hn} + \rho \min\{\mu_{\ell o}, \mu_{hn}\} - \lambda_d\mu_{ho} + \lambda_u\mu_{\ell o}, \tag{3}$$

$$\dot{\mu}_{\ell n} = 2\lambda\mu_{\ell o}\mu_{hn} + \rho \min\{\mu_{\ell o}, \mu_{hn}\} - \lambda_u\mu_{\ell n} + \lambda_d\mu_{hn}. \tag{4}$$

Proposition 1. *There exists a unique equilibrium* $\mu^* = (\mu_{hn}, \mu_{ho}, \mu_{\ell n}, \mu_{\ell o})$ *such that* $\mu^*_{i,j} \in$

$(0,1)$ *for all* i, j.

Proof. Begin by defining the "intensity of demand" to be

$$y = \frac{\lambda_u}{\lambda_u + \lambda_d}.$$

It gives us the proportion of taste shocks that give people a high valuation of the asset.

Take equations (1) and (2) in the steady state so that the LHS are both equal to zero. Equate them, simplify, and rearrange to get

$$\lambda_d[\mu_{ho} + \mu_{hn}] = \lambda_u[\mu_{\ell n} + \mu_{\ell o}].$$

Now add add $\lambda_u[\mu_{ho} + \mu_{hn}]$ to both sides, from which it follows that

$$[\lambda_u + \lambda_d][\mu_{ho} + \mu_{hn}] = \lambda_u \underbrace{[\mu_{ho} + \mu_{\ell o} + \mu_{\ell n} + \mu_{hn}]}_{1}.$$

Notice that we can rewrite the demand intensity indication as $y(\lambda_u + \lambda_d) = \lambda_u$, so plugging this in gives

$$[\mu_{ho} + \mu_{hn}] = y.$$

Makes sense so far: the demand intensity is the high types.

Add and subtract $\mu_{\ell o}$ and then recall the supply condition of $s = \mu_{ho} + \mu_{\ell o}$. Making the substitution yields

$$[s + \mu_{hn} - \mu_{\ell o}] = y \implies \mu_{hn} = \mu_{\ell o} + (y - s). \tag{5}$$

The term $y - s = \mu_{hn} - \mu_{\ell o}$ gives us the difference between buyers and sellers.

We will suppose that $s < y$, i.e. that there are more buyers than sellers. The implication is that $\mu_{\ell o} < \mu_{hn}$. Take equation (3) at the steady state, plug in equation (5), and use $\mu_{ho} = s - \mu_{\ell o}$ to get

$$0 = 2\lambda\mu_{\ell o}^2 + [2\lambda(y - s) + \rho + \lambda_u + \lambda_d]\mu_{\ell o} - s\lambda_d.$$

Define the function

$$Q(\mu_{\ell o}) = 2\lambda\mu_{\ell o}^2 + [2\lambda(y - s) + \rho + \lambda_u + \lambda_d]\mu_{\ell o} - s\lambda_d.$$

Then $Q(\mu_{\ell o}) = 0$ is the solution.

First note that $Q(0) = -s\lambda_d < 0$. Second, note that

$$Q(1) = 2\lambda + 2\lambda(y - s) + \rho + \lambda_u + \lambda_d(1 - s).$$

Since $s < y$, we know that $Q(1) > 0$. Finally, note that

$$Q'(\mu_{\ell o}) = 4\lambda\mu_{\ell o} + 2\lambda(y - s) + \rho + \lambda_u + \lambda_d > 0.$$

So we have an upward sloping line the starts off negative, becomes positive. It must uniquely cross zero once, giving us the unique $\mu^*_{\ell o} \in (0, 1)$ that satisfies $Q(\mu^*_{\ell o}) = 0$.

From the supply condition, it follows that $\mu^*_{ho} = s - \mu^*_{\ell o}$. This is clearly less than 1, but we need to make sure it's greater than zero. Since $Q(\mu^*_{\ell o}) = 0$, we can ensure positivity by showing $Q(s) > 0$. And indeed,

$$Q(s) = 2\lambda s^2 + [2\lambda(y - s) + \rho + \lambda_u]s > 0.$$

Great, so we have unique $\mu^*_{ho} \in (0, 1)$

Now use equation (5) to establish that $\mu^*_{hn} = \mu^*_{\ell o} + (y - s)$. This is clearly greater than zero since $s < y$. But we need to establish that it's less than 1. We need $\mu^*_{\ell o} + (y - s) < 1$, i.e. $\mu^*_{\ell o} < 1 - y + s$. Since the Q function is monotonically increasing, we could also just see if $Q(\mu^*_{\ell o}) = 0 < Q(1 - y + s)$. So let's try it.

$$Q(\mu^*_{\ell o}) = 2\lambda(1 - y + s)^2 + [2\lambda(y - s) + \rho + \lambda_u + \lambda_d](1 - y + s) - s\lambda_d.$$

We'll have two opposite $s\lambda_d$ terms that cancel each other out while the rest is strictly positive. Great, so we have unique $\mu^*_{hn} \in (0, 1)$.

We've pinned down three agent measures, and hence we can simply conclude that $\mu^*_{\ell n} = 1 - \mu^*_{\ell o} - \mu^*_{hn} - \mu^*_{ho}$. □

4 Value Functions

Normalize the high types value of the asset to 1. We will say that the low type valuation is then $1 - \delta$, where δ reflects the difference in valuation.

So the value of being a low owner consists of four parts.

- The value of owning the thing, $1 - \delta$.
- Selling your asset directly to a high owner for price P. The total number of matches

is $2\lambda\mu_{hn}\mu_{\ell o}$. The proportion of matched low types is therefore

$$\frac{2\lambda\mu_{hn}\mu_{\ell o}}{\mu_{\ell o}} = 2\lambda\mu_{hn}.$$

This is the arrival rate of selling the asset to a high type.

- Finding a MM and receiving a payment of B, called the *bid price*, for having them sell your asset. There is ρ chance of hooking up with the MM, and because there are fewer sellers than buyers, we know that all sales can be made. So the arrival rate is simply ρ.
- Getting hit with an "up shock" with arrival rate λ_u.

Putting it all together, the value function is

$$rV_{\ell o} = (1-\delta) + 2\lambda\mu_{hn}[P + V_{\ell n} - V_{\ell o}] + \rho[B + V_{\ell n} - V_{\ell o}] + \lambda_u[V_{ho} - V_{\ell o}]. \tag{6}$$

The story for high-type nonowners is similar, except for two things. First, buyers pay the MM a fee of A, called the *ask price*. Second, there are fewer sellers than buyers. This means that only a fraction of buyers can actually buy, in particular, the proportion $\mu_{\ell o}/\mu_{hn}$. This will be the MM arrival rate. So the value function is

$$rV_{hn} = 2\lambda\mu_{\ell o}[V_{ho} - V_{hn} - P] + \rho\frac{\mu_{\ell o}}{\mu_{hn}}[V_{ho} - V_{hn} - A] + \lambda_d[V_{\ell n} - V_{hn}]. \tag{7}$$

The value functions for the high type owners, who do not want to sell, and the low nonowners, who do not want to buy, are, respectively,

$$rV_{ho} = 1 + \lambda_d[V_{\ell o} - V_{ho}], \tag{8}$$

$$rV_{\ell n} = \lambda_u[V_{hn} - V_{\ell n}]. \tag{9}$$

5 Bargaining

5.1 Investor Bargaining

We'll go with Nash bargaining. Let q be the bargaining power of sellers. Define $\Delta_{V\ell} = V_{\ell o} - V_{\ell n}$ and $\Delta_{Vh} = V_{ho} - V_{hn}$. Then the bargaining problem is

$$\max_P \left\{(P - \Delta_{V\ell})^q(\Delta_{Vh} - P)^{1-q}\right\}.$$

Log it up and take FOC and you'll get

$$\frac{q}{P - \Delta_{V\ell}} = \frac{1-q}{\Delta_{Vh} - P}.$$

From this it follows that

$$P = q\Delta_{Vh} + (1-q)\Delta_{V\ell}. \tag{10}$$

5.2 Market Maker and Low-Type Bargaining

Let z be the MM bargaining power. The interdealer market is Walrasian, so the market maker is a price-taker apropos selling the asset. In particular, the MM receives a price of M for selling the asset in the interdealer market and gives B to the seller. Hence the Nash bargaining problem is

$$\max_B (B - \Delta_{V\ell})^{1-z}(M-B)^z,$$

which yields a solution of

$$B = z\Delta_{V\ell} + (1-z)M. \tag{11}$$

5.3 Market Maker and High-Type Bargaining

In this scenario the MM must buy the asset at price M, then selling it to the high type for A. Thus the bargaining problem is

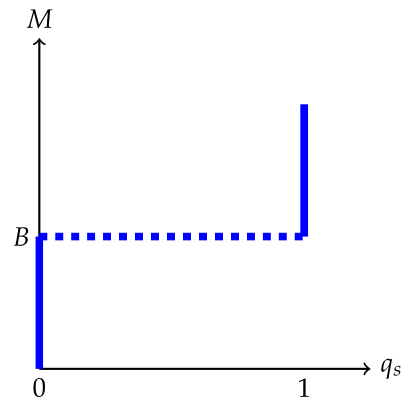
$$\max_A (\Delta_{Vh} - A)^{1-z}(A-M)^z,$$

which yields a solution of

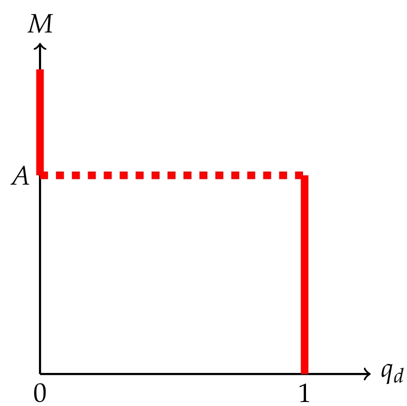
$$A = z\Delta_{Vh} + (1-z)M. \tag{12}$$

6 Interdealer Market

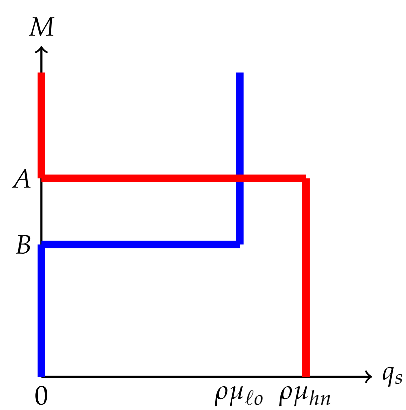
We still gotta figure out what M is in the interdealer market. Think of the individual MM seller. They have one unit they want to sell in order to maximize $q_s[M-B]$, so $q_s \in \{0,1\}$. If $M < B$, they sell nothing. If $M = B$, they're indifferent. If $M > B$. they want to sell $q_s = 1$. So the individual supply curve is



Demand is similar. The buying MM buys $q_d = 1$ if $A > M$. They're indifferent if $A = M$. And they buy nothing if $A < M$.



We can solve this by aggregating. The measure of selling MM are the ones who lumped into low-type owners, and there are $\rho\mu_{\ell o}$ of them. Similarly, there are $\rho\mu_{hn}$ buyers. Hence the market supply and demand curves will be



We can see that the seller side of the market determines both the equilibrium price $M^* = A$ and quantity traded $Q^* = \rho\mu_{\ell o}$. This is a consequence of our assumption that the sellers are lower in measure than buyers. This means the buyer MM get screwed since their surplus is zero. It also implies from the bargaining solution that

$$A = z\Delta_{Vh} + (1-z)A \quad \Longrightarrow \quad A = \Delta_{Vh}$$

7 Solving the Model

Let's list what we have so far, simplifying bits and pieces here and there.

$$\begin{aligned}
rV_{\ell o} &= (1-\delta) + 2\lambda\mu_{hn}[P - \Delta_{V\ell}] + \rho[B - \Delta_{V\ell}] + \lambda_u[V_{ho} - V_{\ell o}], && (13)\\
rV_{\ell n} &= \lambda_u[V_{hn} - V_{\ell n}], && (14)\\
rV_{ho} &= 1 + \lambda_d[V_{\ell o} - V_{ho}], && (15)\\
rV_{hn} &= 2\lambda\mu_{\ell o}[A - P] + \lambda_d[V_{\ell n} - V_{hn}], && (16)\\
P &= qA + (1-q)\Delta_{V\ell}, && (17)\\
B &= z\Delta_{V\ell} + (1-z)A. && (18)
\end{aligned}$$

Take (13) minus (14) and (15) minus (16) to get, respectively,

$$\begin{aligned}
r\Delta_{V\ell} &= (1-\delta) + 2\lambda\mu_{hn}[P - \Delta_{V\ell}] + \rho[B - \Delta_{V\ell}] + \lambda_u[A - \Delta_{V\ell}], && (19)\\
rA &= 1 + \lambda_d[\Delta_{V\ell} - A] - 2\lambda\mu_{\ell o}[A - P]. && (20)
\end{aligned}$$

Now take equation (19) and plug in (17) and (18) for P and B to get

$$[r + 2\lambda\mu_{hn}q + (1-z)\rho + \lambda_u]\Delta_{V\ell} = (1-\delta) + [2\lambda\mu_{hn}q + (1-z)\rho + \lambda_u]A. \tag{21}$$

Do the same thing with equation (20) to get

$$[r + 2\lambda\mu_{\ell o}(1-q) + \lambda_d]A = 1 + [2\lambda\mu_{\ell o}(1-q) + \lambda_d]\Delta_{V\ell}. \tag{22}$$

Equations (21) and (22) look pretty similar. Notice that the brackets on each side of both differ only in r. So let's define $x_u \equiv 2\lambda\mu_{hn}q + (1-z)\rho + \lambda_u$ and $x_d \equiv 2\lambda\mu_{\ell o}(1-q) + \lambda_d$. Then we have

$$\begin{aligned}
[r + x_u]\Delta_{V\ell} &= (1-\delta) + Ax_u, && (23)\\
[r + x_d]A &= 1 + \Delta_{V\ell}x_d. && (24)
\end{aligned}$$

Now add them cross-wise to get

$$[r + x_u + x_d]\Delta_{V\ell} = [r + x_i + x_d]A - \delta,$$

and from this it follows that

$$\Delta_{V\ell} = A - \frac{\delta}{[r + x_u + x_d]}. \tag{25}$$

Now plug this back into (24) to get

$$[r + x_d]A = 1 + Ax_d - \frac{\delta x_d}{[r + x_u + x_d]}.$$

Solve for A to get

$$A = \frac{1}{r} - \frac{\delta}{r}\frac{2\lambda\mu_{\ell o}(1-q) + \lambda_d}{[r + 2\lambda\mu_{hn}q + 2\lambda\mu_{\ell o}(1-q) + (1-z)\rho + \lambda_u + \lambda_d]}. \tag{26}$$

The term $1/r$ is the fundamental value, since r is the discount rate. The other term is an illiquidity discount – in order to get people to buy an illiquid asset, they pay less than what they would if it was fully liquid. The illiquidity, of course, is a consequence search and bargaining costs.

Now plug this monster into equation (18), using equation (25), and you get

$$\begin{aligned} B &= z\Delta_{V\ell} + (1-z)A \\ &= A - \frac{\delta}{r}\frac{zr}{[r + x_u + x_d]} \\ &= \frac{1}{r} - \frac{\delta}{r}\frac{2\lambda\mu_{\ell o}(1-q) + \lambda_d + zr}{[r + 2\lambda\mu_{hn}q + 2\lambda\mu_{\ell o}(1-q) + (1-z)\rho + \lambda_u + \lambda_d]}. \end{aligned} \tag{27}$$

Same idea. Make a mental note of the fact that

$$A - B = z\frac{\delta}{[r + x_u + x_d]},$$

because we'll be using it soon.

Okay, now let's try to get P.

$$
\begin{aligned}
P &= qA + (1-q)\left(A - \frac{\delta}{r}\frac{r}{[r + x_u + x_d]}\right) \\
&= \frac{1}{r} - \frac{\delta}{r}\frac{[2\lambda\mu_{\ell o} + r](1-q) + \lambda_d}{[r + 2\lambda\mu_{hn}q + 2\lambda\mu_{\ell o}(1-q) + (1-z)\rho + \lambda_u + \lambda_d]}.
\end{aligned}
$$

No further comment needed.

8 Equilibrium Properties

8.1 Comparative Statics

Proposition 2. *All three prices are increasing with respect to λ, the rate at which investors meet each other.*

I'll omit the proof, but it is intuitive. Easier matching makes the asset more liquid and thus the illiquidity discount is attenuated, leading to higher prices.

Proposition 3. *The spread $A - B$ is decreasing in $z < 1$, decreasing in λ, and decreasing in ρ.*

I will again rely on intuition. Recall that z is MM bargaining power. If the MM has more bargaining power, then we expect the MM to give less B to the seller and take more A from the buyer. If λ goes up, then dealers become less important and so their influence is diminished. If ρ increases, then any particular MM becomes less powerful since there are more outside options for investors.

Proposition 4. *When $z = 1$, the spread $A - B$ is increasing in ρ.*

In this case the dealer has all of the bargaining power and $A - B$ increases with ρ. Think of this as being the monopolistlc market making scenario; even if an investor can find another MM, think of that MM from being an agent in the same MM company who offers the same deal.

Proof. Okay, suppose $z = 1$. Then

$$
A - B = \frac{\delta}{[r + 2\lambda\mu_{hn}q + 2\lambda\mu_{\ell o}(1-q) + \lambda_u + \lambda_d]}.
$$

There ain't no ρ! But μ_{hn} and $\mu_{\ell o}$ are endogenous so those are affected by changes in ρ. Let D denote the messy denominator. Then we can differentiate and write

$$\begin{aligned}\frac{d[A-B]}{d\rho} &= -\frac{\delta}{D^2}\frac{dD}{d\rho} \\ &= -\frac{\delta}{D^2}\left[2\lambda\frac{d\mu_{hn}}{d\rho}q + 2\lambda\frac{\mu_{\ell o}}{d\rho}(1-q)\right].\end{aligned}$$

Now let's rely on some intuition. If it's easier to meet up with a MM, then we expect low-type owners will be able to sell stuff more and hence their measure will go down. That is, $d\mu_{\ell o}/d\rho < 0$. The logic is the same for $d\mu_{hn}/d\rho < 0$.

So we can conclude that the term in the parenthesis is negative. Negative times negative is positive. The end. □

8.2 Limiting Cases

If we consider a purely Walrasian market, we'd expect the right people to hold the assets and the price to be fundamental, i.e.,

$$\begin{aligned}\mu^*_{ho} &= s, \\ \mu^*_{hn} &= y - s, \\ \mu^*_{\ell o} &= 0, \\ \mu^*_{\ell n} &= 1 - y, \\ P^* &= \frac{1}{r}.\end{aligned}$$

So when our frictions are limited away, the model damn well better go to these values as well.

Proposition 5. *Suppose* $0 < q < 1$. *Then as* $\lambda \to \infty$, $\mu \to \mu^*$ *and* $B, A, P \to P^*$.

Proof. Recall that $\mu^*_{\ell p}$ solves

$$Q(\mu^*_{\ell o}) = 2\lambda[\mu^*_{\ell o}]^2 + [2\lambda(y-s) + \rho + \lambda_u + \lambda_d]\mu^*_{\ell o} - s\lambda_d = 0.$$

Divide by λ and you get

$$2[\mu^*_{\ell o}]^2 + 2(y-s)\mu^*_{\ell o} + \frac{\rho + \lambda_u + \lambda_d}{\lambda}\mu^*_{\ell o} - \frac{s\lambda_d}{\lambda} = 0.$$

Blow λ up to infinity and you have

$$2[\mu^*_{\ell o}]^2 + 2(y-s)\mu^*_{\ell o} = 0,$$

for which the only feasible solution is $\mu^*_{\ell o} = 0$. It follows from the supply condition that $\mu^*_{ho} = s$. Then $\mu^*_{hn} = \mu^*_{\ell o} + (y-s)$ implies that $\mu^*_{hn} = y - s$ and thus to sum to 1 we must have $\mu_{\ell n} = 1 - y$.

Now take the price equation and divide the second term by λ/λ to get

$$\frac{1}{r} - \frac{\delta}{r}\frac{\frac{[2\lambda\mu_{\ell o} + r](1-q) + \lambda_d}{\lambda}}{\left[\frac{r + (1-z)\rho + \lambda_u + \lambda_d}{\lambda} + 2\mu_{hn}q + 2\mu_{\ell o}(1-q)\right]}.$$

It's easy to see in the limit that the second term will have zero numerator and finite denominator, and hence we will just be left with fundamental value $1/r$. B and A follow similarly so I will omit them. □

Proposition 6. *Suppose $z \neq 1$. As $\rho \to \infty$, $\mu \to \mu^*$ and $B, A, P \to P^*$.*

The proof is practically the same (just divide in the same ways by ρ instead of λ), so I'll omit it.