Time-Varying Rates of Return

Holding Rate

Suppose a principal accumulates with return rates r_1 in period 1, r_2 in period 2, and r_3 in period 3. The holding rate is the rate $\bar{r}$ such that the end-of-year 3 principal would be the same were the initial principle to be compounded just once at this rate.

For example, if you invest \$1 for three years, your investment will become worth $\$1(1+r_1)(1+r_2)(1+r_3)$. The holding rate satisfies

$$(1+\bar{r}) = (1+r_1)(1+r_2)(1+r_3).$$

Annualized Rate

The annual rate $\underline{r}$ is the constant rate that gives you the same investment when compounded for the same number of periods. For example,

$$(1+\underline{r})^3 = (1+r_1)(1+r_2)(1+r_3).$$

Time Varying Present Value

Cash flow stream $(C_0, C_1, C_2, \ldots)$ with corresponding rates of return $(0, r_1, r_2, \ldots)$ has present value

$$\text{PV} = C_0 + \frac{C_1}{(1+r_1)} + \frac{C_2}{(1+r_1)(1+r_2)} + \ldots$$

It can be useful to convert the compounded terms into annualized terms. We have

$$(1+\underline{r_1}) = (1+r_1),$$
$$(1+\underline{r_2})^2 = (1+r_1)(1+r_2),$$
$$(1+\underline{r_3})^3 = (1+r_1)(1+r_2)(1+r_3),$$

and so on, and therefore

$$\text{PV} = C_0 + \frac{C_1}{(1+\underline{r_1})} + \frac{C_2}{(1+\underline{r_2})^2} + \frac{C_3}{(1+\underline{r_3})^3} + \ldots$$

Inflation Adjustment

Fisher Equation

Cash flows are usually given in nominal terms. Interest rates are sometimes quoted in real terms. The Fisher equation relates real interest rates with nominal via the rate of inflation π. It is given by

$$(1+r_{\text{nominal}}) = (1+r_{\text{real}})(1+\pi).$$

Some intuition:

- $(1+r_{\text{nominal}})$ is the quoted interest rate in terms of *dollars*, hence called *nominal*.
- $(1+r_{\text{real}})$ is the interest rate in terms of a basket of *goods*, hence called *real*.
- $(1+\pi)$ is the rate of inflation that "eats away" at the purchasing power of dollars, and hence the value in terms of goods.

Uncertainty

Expected Value

The expected value of random variable D that can take on values $D(X) = \big(D(x_1), D(x_2), \ldots\big)$ with respective probabilities $P = (p_1, p_2, \ldots)$ is

$$E_P[D] = p_1 D(x_1) + p_2 D(x_2) + \ldots$$

Present Value with Uncertain Interest Rate

Suppose r is a random variable. Then the present value of C one period from now is

$$\text{PV} = \frac{C}{1+E[r]}.$$

In other words, if we invest $C/(1+E[r])$ today, then we *expect* it to turn into C next period (in the probabalistic sense). Notice that PV is a number – it is not random.

Bonds

Promised Rate of Interest

Consider a riskless bond. L is the price of the bond, T is the repayment period, D is the face value/promised value. The promised rate of interest is given by

$$r_{\text{promised}} = \frac{D-L}{L}.$$

Expected Rate of Return

With probability p, the bond does defaults and nothing is paid out – call this state s_D. With probability $1-p$, the bond does not default and face value $C_{s_{ND}}$ is paid out – call this state s_{ND}. Respectively for each state,

$$r_{s_D} = \frac{-L}{L}, \quad r_{s_{ND}} = \frac{C_{s_{ND}} - L}{L}.$$

The expected rate of return is

$$E[r] = p r_{s_D} + (1-p) r_{s_{ND}}.$$

Present Value

Let $r \equiv E[r]$. The present value of a bond with promised value $C_{s_{ND}}$ is

$$\text{PV} = \frac{(1-p)C_{s_{ND}}}{1+r}.$$

A risk-neutral investor's utility depends only on the investment's expected value. Notice that $r_{s_{ND}} > r$, which is called the default premium.

Levered Equity Funding

No neat, clean formulas here : \