Permanent Income Hypothesis

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1 Simple Case

Suppose an individual will live for T periods, starting with initial wealth A_0. In period t, the individual will consume C_t for utility $u(C_t)$, and will earn labor income Y_t. It is assumed that $u'(\cdot) > 0$ and $u''(\cdot) < 0$, and that Y_t is pre-determined and fully-known by the agent.

The individual can save or borrow at an exogenous interest rate, subject to the constraint that debt be repaid in final period T. For simplicity, let us assume that the interest rate is zero, and furthermore that the agent's discount rate is zero. The individual's lifetime utility is therefore given by

$$U = \sum_{t=1}^{T} u(C_t). \tag{1}$$

Since borrowing and saving allows the individual to essentially transfers funds to or from any period, they face a lifetime budget constraint of the form

$$\sum_{t=1}^{T} C_t = A_0 + \sum_{t=1}^{T} Y_t, \tag{2}$$

where A_0 is initial wealth.

The Lagrangian for this problem is

$$\mathcal{L} = \sum_{t=1}^{T} u(C_t) - \lambda \left[\sum_{t=1}^{T} C_t - A_0 - \sum_{t=1}^{T} Y_t \right].$$

Notice that λ is not indexed by t because it is as if there is only one budget constraint, reflecting the entire lifetime income stream. The first-order condition is

$$u'(C_t) = \lambda. \tag{3}$$

We can conclude that $u'(C_1) = u'(C_2) = \ldots = u'(C_T)$, since the first-order condition holds for any C_t. The shape of the utility function allow us to further conclude that $C_1 = C_2 = \ldots = C_T$. Plugging this result into the budget constraint gives

$$C_t = \frac{1}{T}\left[A_0 + \sum_{\tau=1}^{T} Y_\tau\right]. \tag{4}$$

The interpretation is straightforward: consume the same constant fraction of lifetime wealth in each period.

One important implication is that consumption at any given time is determined by lifetime income. If there is a one-time increase in period t income of the amount X, then consumption in period t only increases by X/T. This suggests that the multiplier for, say, a tax rebate or stimulus check is going be rather small on impact, and "spread thinly," so to speak.

Another important implication is that saving, defined as $S_t \equiv Y_t - C_t$, is going to jump around a lot as transitory income Y_t changes, since C will remain constant. This is obvious if one were to write $\Delta S_t = \Delta Y_t - \Delta C_t$, because $\Delta C_t = 0$ requires $\Delta S_t = \Delta Y_t$. In more detail, we can write

$$\begin{aligned} S_t &= Y_t - C_t \\ &= Y_t - \frac{1}{T}\left[A_0 + \sum_{\tau=1}^{T} Y_\tau\right] \\ &= \left[Y_t - \frac{1}{T}\sum_{\tau=1}^{T} Y_\tau\right] - \frac{1}{T}A_0. \end{aligned} \tag{5}$$

The first term of equation (5) is key. It tells us that saving is going to be high when period t income is higher than average income. Makes sense: the above-average income is transferred to periods of below-average income so that consumption can be constant. Saving and borrowing is what allows smooth consumption over time.

2 Random Walk

An individual is not going to know exactly what their income is every single payday for the rest of their life, so let's weaken that assumption. Supposing quadratic utility, the individual in period $t = 1$ will now seek to maximize lifetime expected utility, that is,

$$E_1[U] = E_1\left[\sum_{t=1}^{T}\left(C_t - \frac{a}{2}C_t^2\right)\right],$$

for $a > 0$. The subscript on the expectations operator denotes the period in which the expectations are formed, in this case in the first period. Keep in mind, however, that Y_1 is known in period 1, and thus $E_1[Y_1] = Y_1$ and $E_1[C_1] = C_1$. This will be useful momentarily.

We will also make the assumption that consumption is always in the range such that $u'(\cdot) > 0$. Borrowing and lending has the same structure as before, but since the individual does not know future income (and thus doesn't know future consumption), we apply expectations to the lifetime budget constraint to get

$$\sum_{t=1}^{T} E_1[C_t] = A_0 + \sum_{t=1}^{T} E_1[Y_t]. \tag{6}$$

In words: how much you expect to consume in total over time must equal how much wealth you expect to obtain over time.

The Lagrangian for this problem is

$$\mathcal{L} = E_1\left[\sum_{t=1}^{T}\left(C_t - \frac{a}{2}C_t^2\right)\right] - \lambda\left[E_1[C_t] - A_0 - \sum_{t=1}^{T} E_1[Y_t]\right],$$

which yields first order condition

$$E[1 - aC_t] = \lambda. \tag{7}$$

Like in the simple case, this holds for any t, and therefore we can conclude that $E[C_1] = E[C_2] = \ldots = E[C_T]$. But since $E[C_1] = C_1$ is known at the start, we can go a step further and write $C_1 = E[C_2] = \ldots = E[C_T]$.

Plugging into the budget constraint gives

$$C_1 = \frac{1}{T}\left(A_0 + \sum_{t=1}^{T} E_1[Y_t]\right). \tag{8}$$

Initial consumption is going to be a fraction of expected lifetime wealth, and consumption in subsequent periods is expected to be the same. To that end, we can write

$$C_t = E_{t-1}[C_t] + e_t, \tag{9}$$

where $E_{t-1}[e_t] = 0$. The idea is that our expectation of tomorrow's consumption will have a little bit of random error to it: expectations about the future are not going to perfect all of the time, otherwise they wouldn't be *expectations* in any meaningful sense. Therefore actual C_t will sometimes be a bit different from what was expected: there will be actual changes is consumption over time. On average that error will be zero, so that expectations are correct on average (i.e. expectations are not biased). Which is to say, consumption is smoothed *in expectation*.

We can go a step further. Note that $E_{t-1}[C_t] = C_{t-1}$. In words, because the expectation error is on average zero, we expect tomorrow's consumption to be the same as today's. Therefore we can write

$$C_t = C_{t-1} + e_t, \tag{10}$$

This says that consumption takes a random walk, technically a martingale.[1]

This framework is nice because we can see what determines changes in consumption e_t. To illustrate, consider the change in consumption from period 1 to period 2. First, note that $A_1 = A_0 + Y_1 - C_1$, in other words, wealth inherited from period 1 is the remaining stock of wealth not consumed in period 1. Then from the

[1]Wikipedia says: "In probability theory, a martingale is a sequence of random variables (i.e., a stochastic process) for which, at a particular time, the conditional expectation of the next value in the sequence is equal to the present value, regardless of all prior values."

perspective of period 2, we can shift indices to conclude that

$$\begin{aligned} C_2 &= \frac{1}{T-1}\left(A_1 + \sum_{t=2}^{T} E_2[Y_t]\right) \\ &= \frac{1}{T-1}\left(A_0 + Y_1 - C_1 + \sum_{t=2}^{T} E_2[Y_t]\right). \end{aligned}$$

Define the difference

$$\sum_{t=2}^{T} E_2[Y_t] - \sum_{t=2}^{T} E_1[Y_t] \tag{11}$$

to be information learned between period 1 and period 2. If we add and subtract the second term, we can write

$$C_2 = \frac{1}{T-1}\left[A_0 + Y_1 - C_1 + \sum_{t=2}^{T} E_1[Y_t] + \left(\sum_{t=2}^{T} E_2[Y_t] - \sum_{t=2}^{T} E_1[Y_t]\right)\right].$$

From equation 8, we can write

$$\begin{aligned} A_0 + Y_1 + \sum_{t=2}^{T} E_1[Y_t] &= A_0 + \sum_{t=1}^{T} E_1[Y_t] \\ &= T \times C_1. \end{aligned}$$

Now we can go a step further and write

$$\begin{aligned} C_2 &= \frac{1}{T-1}\left[T \times C_1 - C_1 + \left(\sum_{t=2}^{T} E_2[Y_t] - \sum_{t=2}^{T} E_1[Y_t]\right)\right] \\ &= C_1 + \frac{1}{T-1}\left(\sum_{t=2}^{T} E_2[Y_t] - \sum_{t=2}^{T} E_1[Y_t]\right). \end{aligned} \tag{12}$$

The interpretation is that consumption tomorrow is equal to consumption today adjusted by new information learned today. If we learned that lifetime income goes up, then the term in the parenthesis is positive and hence we consume more in C_2. Notice that the strength with which consumption is adjusted due to new information depends on how many periods are remaining: the consequence of new information is "spread more thinly" when there are more remaining periods over which to spread.

3 Precautionary Saving

Because $E_t[C_{t+1}] = C_t$, we can conclude that $u'(E_t[C_{t+1}]) = u'(C_t)$. Quadratic utility implies that marginal utility is linear, specifically of form $u'(C_t) = 1 - aC_t$. We can conclude that

$$\begin{aligned} E_t[u'(C_{t+1})] &= E_t[1 - aC_{t+1}] \\ &= 1 - aE[C_{t+1}] \\ &= 1 - aC_t \\ &= u'(C_t). \end{aligned}$$

In time t it is trivially true that $E_t[C_t] = C_t$ because C_t is known, from which it follows that $u'(E_t[C_t]) = u'(C_t)$. Thus we can establish that this is a *certainty equivalent economy* because

$$u'(E_t[C_{t+1}]) = E_t[u'(C_{t+1})]. \tag{13}$$

This linearity is what allows us to reduce the Euler equation to the straightforward result $E_t[C_{t+1}] = C_t$.

Marginal utility need not be linear in generality, of course. One such case involves having $u'''(\cdot) > 0$. This implies that marginal utility is convex, which in turn implies that

$$u'(E_t[C_{t+1}]) < E_t[u'(C_{t+1})]$$

by Jensen's inequality. But notice that C_t cannot equal $E_t[C_{t+1}]$ anymore because that would imply $u'(C_t) < E_t[u'(C_{t+1})]$. Hence a marginal reduction in current consumption would increase expected utility: transfer consumption from today to the future. The positive third derivative induces the individual to save more now instead of smooth consumption, which is known as *precautionary saving*.

Furthermore, an increase in uncertainty will cause more precautionary saving. Consider a scenario in which consumption can either be low or high, C_L or C_H

respectively, each with probability 0.5. Put $u'(C_{t+1})$ on the vertical axis, C_{t+1} on the horizontal axis. Now draw the convex marginal utility function, and plot some C_L and C_H. If you connect $u'(C_L)$ and $u'(C_H)$, you'll find that the line lies above the marginal utility line due to convexity. The difference between $E_t[u'(C_{t+1})]$ and $u'(E_t[C_{t+1}])$ is the impetus for precautionary saving, as explained above. Now increase uncertainty by moving C_L lower and C_H higher while preserving the mean. Notice that when you connect those new dots, the line is even higher above the marginal utility curve, and hence precautionary saving is even stronger.

4 Interest Rates

Now suppose that there is some constant real interest rate r. Accordingly, we must discount values when we're considering the lifetime budget constraint, giving

$$\sum_{t=1}^{T} \frac{1}{(1+r)^t} C_t = A_0 + \sum_{t=1}^{T} \frac{1}{(1+r)^t} Y_t. \tag{14}$$

The model starts in period 1, but we will discount to period 0 for the sake of notation.

Having an interest rate without a discount rate is bizarre, so we will introduce that now. We will use CRRA utility, so the lifetime stream of discounted utility is

$$U = \sum_{t=1}^{\infty} \beta^t \frac{C_t^{1-\theta}}{1-\theta},$$

where θ is the risk aversion parameter and β is the discount rate.

The Lagrangian for this problem is

$$\mathcal{L} = \sum_{t=1}^{\infty} \beta^t \frac{C_t^{1-\theta}}{1-\theta} - \lambda \left[\sum_{t=1}^{T} \frac{1}{(1+r)^t} C_t - A_0 - \sum_{t=1}^{T} \frac{1}{(1+r)^t} Y_t \right],$$

giving first order conditions with respect to C_t and C_{t+1} of

$$(1+r)^t \beta^t C_t^{-\theta} = \lambda, \tag{15}$$

$$(1+r)^{t+1} \beta^{t+1} C_{t+1}^{-\theta} = \lambda. \tag{16}$$

Equating the two yields the Euler equation

$$\frac{C_{t+1}}{C_t} = [\beta(1+r)]^{1/\theta}. \tag{17}$$

Notice that consumption is only smoothed when $\beta(1+r) = 1$. Now hold β fixed and increase r so that $\beta(1+r) > 1$. The preceding equation implies that consumption grows over time: the interest rate is high, so the individual chooses to save more early on in order to take advantage of that high interest rate. You could also hold r fixed and increase β: the individual emphasizes future well-being when β is larger, so the individual chooses to save more early on in order to boost future consumption. Similar interpretations follow if $\beta(1+r) < 1$.